THE ELEMENTS

OF

COORDINATE GEOMETRY.

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THE STRAIGHT LINE AND CIRCLE

BY

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PREFACE.

IN the following work I have tried to present the elements of Coordinate Geometry in a manner suitable for Beginners and Junior Students. The present book only deals with Cartesian and Polar Coordinates. Within these limits I venture to hope that the book is fairly complete, and that no propositions of very great importance have been omitted.

The Straight Line and Circle have been treated more fully than the other portions of the subject, since it is generally in the elementary conceptions that beginners find great difficulties.

There are a large number of Examples, over 1100 in all, and they are, in general, of an elementary character. The examples are especially numerous in the earlier parts of the book.

I am much indebted to several friends for reading portions of the proof sheets, but especially to Mr W J. Dobbs, M.A. who has kindly read the whole of the book and made many valuable suggestions.

For any criticisms, suggestions, or corrections, I shall be grateful.

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CHAPTER I.

INTRODUCTION.

SOME ALGEBRAIC RESULTS.

1. Quadratic Equations. The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

may easily be shewn to be

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

They are therefore real and unequal, equal, or imaginary, according as the quantity $b^2 - 4ac$ is positive, zero, or negative,

$$\text{i.e. according as } b^2 \gtreqless 4ac.$$

2. *Relations between the roots of any algebraic equation and the coefficients of the terms of the equation.*

If any equation be written so that the coefficient of the highest term is unity, it is shewn in any treatise on Algebra that

(1) the sum of the roots is equal to the coefficient of the second term with its sign changed,

(2) the sum of the products of the roots, taken two at a time, is equal to the coefficient of the third term,

(3) the sum of their products, taken three at a time, is equal to the coefficient of the fourth term with its sign changed,

and so on.

Ex. 1. If α and β be the roots of the equation

$$ax^2+bx+c=0, \text{ i.e. } x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

we have
$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Ex. 2. If α, β, and γ be the roots of the cubic equation

$$ax^3+bx^2+cx+d=0,$$

i.e. of
$$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0,$$

we have
$$\alpha+\beta+\gamma=-\frac{b}{a},$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\frac{c}{a},$$

and
$$\alpha\beta\gamma=-\frac{d}{a}$$

3. It can easily be shewn that the solution of the equations

$$a_1x+b_1y+c_1z=0,$$

and
$$a_2x+b_2y+c_2z=0,$$

is
$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Determinant Notation.

4. The quantity $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for the quantity $a_1b_2-a_2b_1$, so that

$$\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}=a_1b_2-a_2b_1.$$

Exs. (i) $\begin{vmatrix} 2, & 3 \\ 4, & 5 \end{vmatrix}=2\times5-4\times3=10-12=-2;$

(ii) $\begin{vmatrix} -3, & -4 \\ -7, & -6 \end{vmatrix}=-3\times(-6)-(-7)\times(-4)=18-28=-10.$

5. The quantity $$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
is called a determinant of the third order and stands for the quantity
$$a_1 \times \begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1, & b_2 \\ c_1, & c_2 \end{vmatrix} \ldots\ldots\ldots (2),$$
i.e. by Art. 4, for the quantity
$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1),$$
i.e. $$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

6. A determinant of the third order is therefore reduced to three determinants of the second order by the following rule:

Take in order the quantities which occur in the first row of the determinant; multiply each of these in turn by the determinant which is obtained by erasing the row and column to which it belongs; prefix the sign + and − alternately to the products thus obtained and add the results.

Thus, if in (1) we omit the row and column to which a_1 belongs, we have left the determinant $\begin{vmatrix} b_2, & b_3 \\ c_2, & c_3 \end{vmatrix}$ and this is the coefficient of a_1 in (2).

Similarly, if in (1) we omit the row and column to which a_2 belongs, we have left the determinant $\begin{vmatrix} b_1, & b_3 \\ c_1, & c_3 \end{vmatrix}$ and this with the − sign prefixed is the coefficient of a_2 in (2).

7. Ex. The determinant $\begin{vmatrix} 1, & -2, & -3 \\ -4, & 5, & -6 \\ -7, & 8, & -9 \end{vmatrix}$
$$= 1 \times \begin{vmatrix} 5, & -6 \\ 8, & -9 \end{vmatrix} - (-2) \times \begin{vmatrix} -4, & -6 \\ -7, & -9 \end{vmatrix} + (-3) \times \begin{vmatrix} -4, & 5 \\ -7, & 8 \end{vmatrix}$$
$$= \{5 \times (-9) - 8 \times (-6)\} + 2 \times \{(-4)(-9) - (-7)(-6)\}$$
$$- 3 \times \{(-4) \times 8 - (-7) \times 5\}$$
$$= \{-45 + 48\} + 2\{36 - 42\} - 3\{-32 + 35\}$$
$$= 3 - 12 - 9 = -18.$$

8. The quantity $\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}$

is called a determinant of the fourth order and stands for the quantity

$$a_1 \times \begin{vmatrix} b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \\ d_2, & d_3, & d_4 \end{vmatrix} - a_2 \times \begin{vmatrix} b_1, & b_3, & b_4 \\ c_1, & c_3, & c_4 \\ d_1, & d_3, & d_4 \end{vmatrix}$$
$$+ a_3 \times \begin{vmatrix} b_1, & b_2, & b_4 \\ c_1, & c_2, & c_4 \\ d_1, & d_2, & d_4 \end{vmatrix} - a_4 \times \begin{vmatrix} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix},$$

and its value may be obtained by finding the value of each of these four determinants by the rule of Art. 6.

The rule for finding the value of a determinant of the fourth order in terms of determinants of the third order is clearly the same as that for one of the third order given in Art. 6.

Similarly for determinants of higher orders.

9. A determinant of the second order has two terms. One of the third order has 3×2, *i.e.* 6, terms. One of the fourth order has $4 \times 3 \times 2$, *i.e.* 24, terms, and so on.

10. Exs. Prove that

(1) $\begin{vmatrix} 2, & -3 \\ 4, & 8 \end{vmatrix} = 28.$ (2) $\begin{vmatrix} -6, & 7 \\ -4, & -9 \end{vmatrix} = 82.$ (3) $\begin{vmatrix} 5, & -3, & 7 \\ -2, & 4, & -8 \\ 9, & 3, & -10 \end{vmatrix} = -98.$

(4) $\begin{vmatrix} 9, & 8, & 7 \\ 6, & 5, & 4 \\ 3, & 2, & 1 \end{vmatrix} = 0.$ (5) $\begin{vmatrix} -a, & b, & c \\ a, & -b, & c \\ a, & b, & -c \end{vmatrix} = 4abc.$

(6) $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$

Elimination.

11. Suppose we have the two equations

$$a_1x + a_2y = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$b_1x + b_2y = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

between the two unknown quantities x and y. There must be some relation holding between the four coefficients a_1, a_2, b_1, and b_2. For, from (1), we have

$$\frac{x}{y} = -\frac{a_2}{a_1},$$

and, from (2), we have $\dfrac{x}{y} = -\dfrac{b_2}{b_1}$.

Equating these two values of $\dfrac{x}{y}$ we have

$$\frac{b_2}{b_1} = \frac{a_2}{a_1},$$

i.e. $$a_1b_2 - a_2b_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The result (3) is the condition that both the equations (1) and (2) should be true for the same values of x and y. The process of finding this condition is called the eliminating of x and y from the equations (1) and (2), and the result (3) is often called the eliminant of (1) and (2).

Using the notation of Art. 4, the result (3) may be written in the form $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} = 0.$

This result is obtained from (1) and (2) by taking the coefficients of x and y in the order in which they occur in the equations, placing them in this order to form a determinant, and equating it to zero.

12. Suppose, again, that we have the three equations

$$a_1x + a_2y + a_3z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$b_1x + b_2y + b_3z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and $$c_1x + c_2y + c_3z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

between the three unknown quantities x, y, and z.

By dividing each equation by z we have three equations between the two unknown quantities $\frac{x}{z}$ and $\frac{y}{z}$. Two of these will be sufficient to determine these quantities. By substituting their values in the third equation we shall obtain a relation between the nine coefficients.

Or we may proceed thus. From the equations (2) and (3) we have

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{b_3c_1 - b_1c_3} = \frac{z}{b_1c_2 - b_2c_1}.$$

Substituting these values in (1), we have

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0 \ldots (4).$$

This is the result of eliminating x, y, and z from the equations (1), (2), and (3).

But, by Art. 5, equation (4) may be written in the form

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0.$$

This eliminant may be written down as in the last article, *viz.* by taking the coefficients of x, y, and z in the order in which they occur in the equations (1), (2), and (3), placing them to form a determinant, and equating it to zero.

13. Ex. *What is the value of a so that the equations*

$$ax + 2y + 3z = 0, \quad 2x - 3y + 4z = 0,$$

and

$$5x + 7y - 8z = 0$$

may be simultaneously true?

Eliminating x, y, and z, we have

$$\begin{vmatrix} a, & 2, & 3 \\ 2, & -3, & 4 \\ 5, & 7, & -8 \end{vmatrix} = 0,$$

i.e. $a[(-3)(-8) - 4 \times 7] - 2[2 \times (-8) - 4 \times 5] + 3[2 \times 7 - 5 \times (-3)] = 0$,

i.e. $a[-4] - 2[-36] + 3[29] = 0$,

so that $a = \frac{72 + 87}{4} = \frac{159}{4}$.

14. If again we have the four equations

$$a_1x + a_2y + a_3z + a_4u = 0,$$
$$b_1x + b_2y + b_3z + b_4u = 0,$$
$$c_1x + c_2y + c_3z + c_4u = 0,$$
and
$$d_1x + d_2y + d_3z + d_4u = 0,$$

it could be shewn that the result of eliminating the four quantities x, y, z, and u is the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = 0.$$

A similar theorem could be shewn to be true for n equations of the first degree, such as the above, between n unknown quantities.

It will be noted that the right-hand member of each of the above equations is zero.

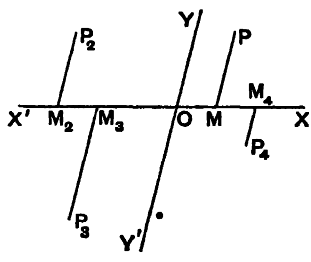
CHAPTER II.

COORDINATES. LENGTHS OF STRAIGHT LINES AND AREAS OF TRIANGLES.

15. Coordinates. Let OX and OY be two fixed straight lines in the plane of the paper. The line OX is called the axis of x, the line OY the axis of y, whilst the two together are called the axes of coordinates.

The point O is called the origin of coordinates or, more shortly, the origin.

From any point P in the plane draw a straight line parallel to OY to meet OX in M.



The distance OM is called the Abscissa, and the distance MP the Ordinate of the point P, whilst the abscissa and the ordinate together are called its Coordinates.

Distances measured parallel to OX are called x, with or without a suffix, (*e.g.* x_1, x_2... x', x'',..), and distances measured parallel to OY are called y, with or without a suffix, (*e.g.* y_1, y_2,... y', y'',...).

If the distances OM and MP be respectively x and y, the coordinates of P are, for brevity, denoted by the symbol (x, y).

Conversely, when we are given that the coordinates of a point P are (x, y) we know its position. For from O we have only to measure a distance OM $(=x)$ along OX and

then from M measure a distance MP $(=y)$ parallel to OY and we arrive at the position of the point P. For example in the figure, if OM be equal to the unit of length and $MP=2OM$, then P is the point (1, 2).

16. Produce XO backwards to form the line OX' and YO backwards to become OY'. In Analytical Geometry we have the same rule as to signs that the student has already met with in Trigonometry.

Lines measured parallel to OX are positive whilst those measured parallel to OX' are negative; lines measured parallel to OY are positive and those parallel to OY' are negative.

If P_2 be in the quadrant YOX' and P_2M_2, drawn parallel to the axis of y, meet OX' in M_2, and if the numerical values of the quantities OM_2 and M_2P_2 be a and b, the coordinates of P are ($-a$ and b) and the position of P_2 is given by the symbol $(-a, b)$.

Similarly, if P_3 be in the third quadrant $X'OY'$, both of its coordinates are negative, and, if the numerical lengths of OM_3 and M_3P_3 be c and d, then P_3 is denoted by the symbol $(-c, -d)$.

Finally, if P_4 lie in the fourth quadrant its abscissa is positive and its ordinate is negative.

17. Ex. *Lay down on paper the position of the points*

(i) $(2, -1)$, (ii) $(-3, 2)$, and (iii) $(-2, -3)$.

To get the first point we measure a distance 2 along OX and then a distance 1 parallel to OY'; we thus arrive at the required point.

To get the second point, we measure a distance 3 along OX', and then 2 parallel to OY.

To get the third point, we measure 2 along OX' and then 3 parallel to OY'.

These three points are respectively the points P_4, P_2, and P_3 in the figure of Art. 15.

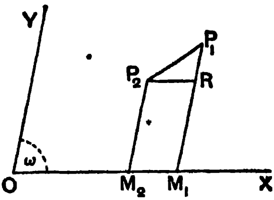
18. When the axes of coordinates are as in the figure of Art. 15, not at right angles, they are said to be Oblique Axes, and the angle between their two positive directions OX and OY, *i.e.* the angle XOY, is generally denoted by the Greek letter ω.

In general, it is however found to be more convenient to take the axes OX and OY at right angles. They are then said to be **Rectangular Axes.**

It may always be assumed throughout this book that the axes are rectangular unless it is otherwise stated.

19. The system of coordinates spoken of in the last few articles is known as the Cartesian System of Coordinates. It is so called because this system was first introduced by the philosopher Des Cartes. There are other systems of coordinates in use, but the Cartesian system is by far the most important.

20. *To find the distance between two points whose coordinates are given.*



Let P_1 and P_2 be the two given points, and let their coordinates be respectively (x_1, y_1) and (x_2, y_2).

Draw P_1M_1 and P_2M_2 parallel to OY, to meet OX in M_1 and M_2. Draw P_2R parallel to OX to meet M_1P_1 in R.

Then

$$P_2R = M_2M_1 = OM_1 - OM_2 = x_1 - x_2,$$

$$RP_1 = M_1P_1 - M_2P_2 = y_1 - y_2,$$

and $\quad \angle P_2RP_1 = \angle OM_1P_1 = 180^\circ - P_1M_1X = 180^\circ - \omega.$

We therefore have [*Trigonometry*, Art. 164]

$$P_1P_2^2 = P_2R^2 + RP_1^2 - 2P_2R \,.\, RP_1 \cos P_2RP_1$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 - 2(x_1 - x_2)(y_1 - y_2)\cos(180^\circ - \omega)$$

$$= \mathbf{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega} \ldots (1).$$

If the axes be, as is generally the case, at right angles, we have $\omega = 90^\circ$ and hence $\cos\omega = 0$.

The formula (1) then becomes

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

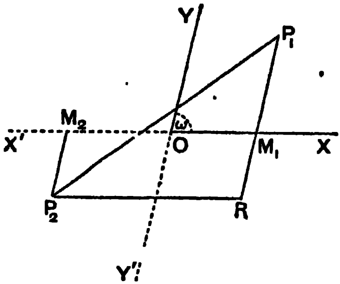
so that in rectangular coordinates the distance between the two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ldots\ldots\ldots\ldots (2).$$

Cor. The distance of the point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$, the axes being rectangular. This follows from (2) by making both x_2 and y_2 equal to zero.

21. The formula of the previous article has been proved for the case when the coordinates of both the points are all positive.

Due regard being had to the signs of the coordinates, the formula will be found to be true for all points.



As a numerical example, let P_1 be the point (5, 6) and P_2 be the point (−7, −4), so that we have

$$x_1 = 5,\ y_1 = 6,\ x_2 = -7,$$

and $$y_2 = -4.$$

Then

$$P_2R = M_2O + OM_1 = 7 + 5$$
$$= -x_2 + x_1,$$

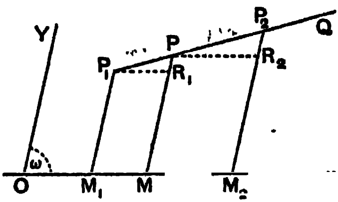
and

$$RP_1 = RM_1 + M_1P_1 = 4 + 6$$
$$= -y_2 + y_1.$$

The rest of the proof is as in the last article.

Similarly any other case could be considered.

22. *To find the coordinates of the point which divides in a given ratio $(m_1 : m_2)$ the line joining two given points (x_1, y_1) and (x_2, y_2).*



Let P_1 be the point (x_1, y_1), P_2 the point (x_2, y_2), and P the required point, so that we have

$$P_1P : PP_2 :: m_1 : m_2.$$

Let P be the point (x, y) so that if P_1M_1, PM, and P_2M_2 be drawn parallel to the axis of y to meet the axis of x in M_1, M, and M_2, we have

$$OM_1 = x_1,\ \ M_1P_1 = y_1,\ \ OM = x,\ \ MP = y,\ \ OM_2 = x_2,$$

and $$M_2P_2 = y_2.$$

Draw P_1R_1 and PR_2, parallel to OX, to meet MP and M_2P_2 in R_1 and R_2 respectively.

Then $$P_1R_1 = M_1M = OM - OM_1 = x - x_1,$$

$$PR_2 = MM_2 = OM_2 - OM = x_2 - x,$$

$$R_1P = MP - M_1P_1 = y - y_1,$$

and $$R_2P_2 = M_2P_2 - MP = y_2 - y.$$

From the similar triangles P_1R_1P and PR_2P_2 we have

$$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{P_1R_1}{PR_2} = \frac{x - x_1}{x_2 - x}.$$

$$\therefore\ m_1(x_2 - x) = m_2(x - x_1);$$

i.e. $$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}.$$

Again $$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{R_1P}{R_2P_2} = \frac{y - y_1}{y_2 - y},$$

so that $$m_1(y_2 - y) = m_2(y - y_1),$$

and hence $$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

The coordinates of the point which divides P_1P_2 internally in the given ratio $m_1 : m_2$ are therefore

$$\frac{\mathbf{m_1x_2 + m_2x_1}}{\mathbf{m_1 + m_2}} \text{ and } \frac{\mathbf{m_1y_2 + m_2y_1}}{\mathbf{m_1 + m_2}}.$$

If the point Q divide the line P_1P_2 *externally* in the same ratio, *i.e.* so that $P_1Q : QP_2 :: m_1 : m_2$, its coordinates would be found to be

$$\frac{\mathbf{m_1x_2 - m_2x_1}}{\mathbf{m_1 - m_2}} \text{ and } \frac{\mathbf{m_1y_2 - m_2y_1}}{\mathbf{m_1 - m_2}}.$$

The proof of this statement is similar to that of the preceding article and is left as an exercise for the student.

Cor. The coordinates of the middle point of the line joining (x_1, y_1) to (x_2, y_2) are

$$\frac{x_1 + x_2}{2} \text{ and } \frac{y_1 + y_2}{2}.$$

23. Ex. 1. *In any triangle ABC prove that*

$$AB^2 + AC^2 = 2(AD^2 + DC^2),$$

where D is the middle point of BC.

Take B as origin, BC as the axis of x, and a line through B perpendicular to BC as the axis of y.

Let $BC = a$, so that C is the point (a, O), and let A be the point (x_1, y_1).

Then D is the point $\left(\frac{a}{2}, O\right)$.

Hence $\qquad AD^2 = \left(x_1 - \frac{a}{2}\right)^2 + y_1^2$, and $DC^2 = \left(\frac{a}{2}\right)^2$.

Hence $\qquad 2(AD^2 + DC^2) = 2\left[x_1^2 + y_1^2 - ax_1 + \frac{a^2}{2}\right]$

$$= 2x_1^2 + 2y_1^2 - 2ax_1 + a^2.$$

Also $\qquad AC^2 = (x_1 - a)^2 + y_1^2,$

and $\qquad AB^2 = x_1^2 + y_1^2.$

Therefore $\qquad AB^2 + AC^2 = 2x_1^2 + 2y_1^2 - 2ax_1 + a^2.$

Hence $\qquad AB^2 + AC^2 = 2(AD^2 + DC^2).$

Ex. 2. *ABC is a triangle and D, E, and F are the middle points of the sides BC, CA, and AB; prove that the point which divides AD internally in the ratio 2 : 1 also divides the lines BE and CF in the same ratio.*

Hence prove that the medians of a triangle meet in a point.

Let the coordinates of the vertices A, B, and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively.

The coordinates of D are therefore $\frac{x_2 + x_3}{2}$ and $\frac{y_2 + y_3}{2}$.

Let G be the point that divides internally AD in the ratio $2 : 1$, and let its coordinates be $\bar{x}$ and $\bar{y}$.

By the last article

$$\bar{x} = \frac{2 \times \frac{x_2 + x_3}{2} + 1 \times x_1}{2 + 1} = \frac{x_1 + x_2 + x_3}{3}.$$

So $\qquad \bar{y} = \frac{y_1 + y_2 + y_3}{3}.$

In the same manner we could shew that these are the coordinates of the points that divide BE and CF in the ratio $2:1$.

Since the point whose coordinates are

$$\frac{x_1+x_2+x_3}{3} \text{ and } \frac{y_1+y_2+y_3}{3}$$

lies on each of the lines AD, BE, and CF, it follows that these three lines meet in a point.

This point is called the Centroid of the triangle.

EXAMPLES. I.

Find the distances between the following pairs of points.

1. $(2, 3)$ and $(5, 7)$.
2. $(4, -7)$ and $(-1, 5)$.
3. $(-3, -2)$ and $(-6, 7)$, the axes being inclined at $60°$.
4. (a, o) and (o, b).
5. $(b+c, c+a)$ and $(c+a, a+b)$.
6. $(a \cos\alpha, a \sin\alpha)$ and $(a \cos\beta, a \sin\beta)$.
7. $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$.
8. Lay down in a figure the positions of the points $(1, -3)$ and $(-2, 1)$, and prove that the distance between them is 5.
9. Find the value of x_1 if the distance between the points $(x_1, 2)$ and $(3, 4)$ be 8.
10. A line is of length 10 and one end is at the point $(2, -3)$; if the abscissa of the other end be 10, prove that its ordinate must be 3 or -9.
11. Prove that the points $(2a, 4a)$, $(2a, 6a)$, and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose side is $2a$.
12. Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are at the vertices of a parallelogram.
13. Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$, and $(-1, 1)$ are at the angular points of a rectangle.
14. Prove that the point $(-\frac{1}{14}, \frac{39}{14})$ is the centre of the circle circumscribing the triangle whose angular points are $(1, 1)$, $(2, 3)$, and $(-2, 2)$.

Find the coordinates of the point which

15. divides the line joining the points $(1, 3)$ and $(2, 7)$ in the ratio $3:4$.
16. divides the same line in the ratio $3:-4$.
17. divides, internally and externally, the line joining $(-1, 2)$ to $(4, -5)$ in the ratio $2:3$.

18. divides, internally and externally, the line joining $(-8, -4)$ to $(-8, 7)$ in the ratio $7 : 5$.

19. The line joining the points $(1, -2)$ and $(-3, 4)$ is trisected; find the coordinates of the points of trisection.

20. The line joining the points $(-6, 8)$ and $(8, -6)$ is divided into four equal parts; find the coordinates of the points of section.

21. Find the coordinates of the points which divide, internally and externally, the line joining the point $(a+b, a-b)$ to the point $(a-b, a+b)$ in the ratio $a : b$.

22. The coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). The line joining the first two is divided in the ratio $l : k$, and the line joining this point of division to the opposite angular point is then divided in the ratio $m : k+l$. Find the coordinates of the latter point of section.

23. Prove that the coordinates, x and y, of the middle point of the line joining the point $(2, 3)$ to the point $(3, 4)$ satisfy the equation

$$x - y + 1 = 0.$$

24. If G be the centroid of a triangle ABC and O be any other point, prove that

$$3(GA^2 + GB^2 + GC^2) = BC^2 + CA^2 + AB^2,$$

and $$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

25. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.

26. $A, B, C, D\ldots$ are n points in a plane whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3),.... AB is bisected in the point G_1; G_1C is divided at G_2 in the ratio $1 : 2$; G_2D is divided at G_3 in the ratio $1 : 3$; G_3E at G_4 in the ratio $1 : 4$, and so on until all the points are exhausted. Shew that the coordinates of the final point so obtained are

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \text{ and } \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n}.$$

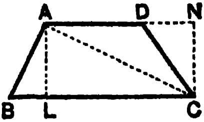
[This point is called the **Centre of Mean Position** of the n given points.]

27. Prove that a point can be found which is at the same distance from each of the four points

$$\left(am_1, \frac{a}{m_1}\right), \left(am_2, \frac{a}{m_2}\right), \left(am_3, \frac{a}{m_3}\right), \text{ and } \left(\frac{a}{m_1 m_2 m_3}, am_1 m_2 m_3\right).$$

24. *To prove that the area of a trapezium,* i.e. *a quadrilateral having two sides parallel, is one half the sum of the two parallel sides multiplied by the perpendicular distance between them.*

Let $ABCD$ be the trapezium having the sides AD and BC parallel.

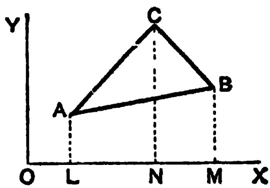


Join AC and draw AL perpendicular to BC and CN perpendicular to AD, produced if necessary.

Since the area of a triangle is one half the product of any side and the perpendicular drawn from the opposite angle, we have

$$\begin{aligned}\text{area } ABCD &= \Delta ABC + \Delta ACD\\ &= \tfrac{1}{2} . BC . AL + \tfrac{1}{2} . AD . CN\\ &= \tfrac{1}{2}(BC + AD) \times AL.\end{aligned}$$

25. *To find the area of the triangle, the coordinates of whose angular points are given, the axes being rectangular.*



Let ABC be the triangle and let the coordinates of its angular points A, B and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Draw AL, BM, and CN perpendicular to the axis of x, and let Δ denote the required area.

Then

$\Delta =$ trapezium $ALNC$ + trapezium $CNMB$ − trapezium $ALMB$
$= \tfrac{1}{2} LN(LA + NC) + \tfrac{1}{2} NM(NC + MB) - \tfrac{1}{2} LM(LA + MB),$
by the last article,

$$= \tfrac{1}{2}\left[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)\right].$$

On simplifying we easily have

$$\boldsymbol{\Delta = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)},$$

or the equivalent form

$$\Delta = \tfrac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

If we use the determinant notation this may be written (as in Art. 5)

$$\Delta = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1\\ x_2, & y_2, & 1\\ x_3, & y_3, & 1\end{vmatrix}.$$

Cor. The area of the triangle whose vertices are the origin $(0, 0)$ and the points (x_1, y_1), (x_2, y_2) is $\tfrac{1}{2}(x_1y_2 - x_2y_1)$.

26. In the preceding article, if the axes be oblique, the perpendiculars AL, BM, and CN, are not equal to the ordinates y_1, y_2, and y_3, but are equal respectively to $y_1 \sin \omega$, $y_2 \sin \omega$, and $y_3 \sin \omega$.

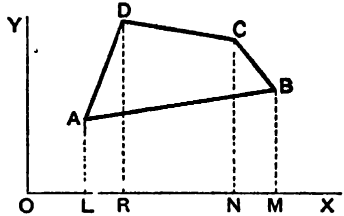
The area of the triangle in this case becomes

$$\tfrac{1}{2}\sin\omega\,\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\},$$

$$\tfrac{1}{2}\sin\omega \times \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

27. In order that the expression for the area in Art. 25 may be a positive quantity (as all areas necessarily are) the points A, B, and C must be taken in the order in which they would be met by a person starting from A and walking round the triangle in such a manner that *the area of the triangle is always on his left hand*. Otherwise the expressions of Art. 25 would be found to be negative.

28. *To find the area of a quadrilateral the coordinates of whose angular points are given.*



Let the angular points of the quadrilateral, taken in order, be A, B, C, and D, and let their coordinates be respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4).

Draw AL, BM, CN, and DR perpendicular to the axis of x.

Then the area of the quadrilateral

$$= \text{trapezium } ALRD + \text{trapezium } DRNC + \text{trapezium } CNMB - \text{trapezium } ALMB$$

$$= \tfrac{1}{2}LR\,(LA + RD) + \tfrac{1}{2}RN\,(RD + NC) + \tfrac{1}{2}NM\,(NC + MB) - \tfrac{1}{2}LM\,(LA + MB)$$

$$= \tfrac{1}{2}\{(x_4 - x_1)(y_1 + y_4) + (x_3 - x_4)(y_3 + y_4) + (x_2 - x_3)(y_3 + y_2) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}.$$

29. The above formula may also be obtained by drawing the lines OA, OB, OC and OD. For the quadrilateral $ABCD$

$$= \triangle OBC + \triangle OCD - \triangle OBA - \triangle OAD.$$

But the coordinates of the vertices of the triangle OBC are $(0, 0)$, (x_2, y_2) and (x_3, y_3); hence, by Art. 25, its area is $\frac{1}{2}(x_2y_3 - x_3y_2)$.

So for the other triangles.

The required area therefore

$$= \tfrac{1}{2}\left[(x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) - (x_2y_1 - x_1y_2) - (x_1y_4 - x_4y_1)\right]$$
$$= \tfrac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\right].$$

In a similar manner it may be shewn that the area of a polygon of n sides the coordinates of whose angular points, taken in order, are

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3), \ldots (x_n, y_n)$$

is $$\tfrac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots + (x_ny_1 - x_1y_n)\right].$$

EXAMPLES. II.

Find the areas of the triangles the coordinates of whose angular points are respectively

1. $(1, 3)$, $(-7, 6)$ and $(5, -1)$. 2. $(0, 4)$, $(3, 6)$ and $(-8, -2)$.
3. $(5, 2)$, $(-9, -3)$ and $(-3, -5)$.
4. $(a, b+c)$, $(a, b-c)$ and $(-a, c)$.
5. $(a, c+a)$, (a, c) and $(-a, c-a)$.
6. $(a\cos\phi_1, b\sin\phi_1)$, $(a\cos\phi_2, b\sin\phi_2)$ and $(a\cos\phi_3, b\sin\phi_3)$.
7. $(am_1^2, 2am_1)$, $(am_2^2, 2am_2)$ and $(am_3^2, 2am_3)$.
8. $\{am_1m_2, a(m_1+m_2)\}$, $\{am_2m_3, a(m_2+m_3)\}$ and $\{am_3m_1, a(m_3+m_1)\}$.
9. $\left\{am_1, \dfrac{a}{m_1}\right\}$, $\left\{am_2, \dfrac{a}{m_2}\right\}$ and $\left\{am_3, \dfrac{a}{m_3}\right\}$.

Prove (by shewing that the area of the triangle formed by them is *zero*) that the following sets of three points are in a straight line:

10. $(1, 4)$, $(3, -2)$, and $(-3, 16)$.
11. $(-\frac{1}{2}, 3)$, $(-5, 6)$, and $(-8, 8)$.
12. $(a, b+c)$, $(b, c+a)$, and $(c, a+b)$.

Find the areas of the quadrilaterals the coordinates of whose angular points, taken in order, are

13. $(1, 1)$, $(3, 4)$, $(5, -2)$, and $(4, -7)$.

14. $(-1, 6)$, $(-3, -9)$, $(5, -8)$, and $(3, 9)$.

15. If O be the origin, and if the coordinates of any two points P_1 and P_2 be respectively (x_1, y_1) and (x_2, y_2), prove that

$$OP_1 . OP_2 . \cos P_1OP_2 = x_1x_2 + y_1y_2.$$

30. Polar Coordinates. There is another method, which is often used, for determining the position of a point in a plane.

Suppose O to be a fixed point, called the **origin** or **pole**, and OX a fixed line, called the **initial line.**

Take any other point P in the plane of the paper and join OP. The position of P is clearly known when the angle XOP and the length OP are given.

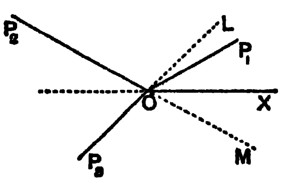
[For giving the angle XOP shews the direction in which OP is drawn, and giving the distance OP tells the distance of P along this direction.]

The angle XOP which would be traced out by the line OP in revolving from the initial line OX is called the vectorial angle of P and the length OP is called its radius vector. The two taken together are called the polar coordinates of P.

If the vectorial angle be θ and the radius vector be r, the position of P is denoted by the symbol (r, θ).

The radius vector is positive if it be measured from the origin O along the line bounding the vectorial angle; if measured in the opposite direction it is negative.

31. Ex. *Construct the positions of the points* (i) $(2, 30°)$, (ii) $(3, 150°)$, (iii) $(-2, 45°)$, (iv) $(-3, 330°)$, (v) $(3, -210°)$ *and* (vi) $(-3, -30°)$.



(i) To construct the first point, let the radius vector revolve from OX through an angle of $30°$, and then mark off along it a distance equal to two units of length. We thus obtain the point P_1.

(ii) For the second point, the radius vector revolves from OX through $150°$ and is then in the position OP_2; measuring a distance 3 along it we arrive at P_2.

(iii) For the third point, let the radius vector revolve from OX through 45° into the position OL. We have now to measure along OL a distance -2, *i.e.* we have to measure a distance 2 not along OL but in the *opposite* direction. Producing LO to P_3, so that OP_3 is 2 units of length, we have the required point P_3.

(iv) To get the fourth point, we let the radius vector rotate from OX through 330° into the position OM and measure on it a distance -3, *i.e.* 3 in the direction MO produced. We thus have the point P_2, which is the same as the point given by (ii).

(v) If the radius vector rotate through $-210°$, it will be in the position OP_2, and the point required is P_2.

(vi) For the sixth point, the radius vector, after rotating through $-30°$, is in the position OM. We then measure -3 along it, *i.e.* 3 in the direction MO produced, and once more arrive at the point P_2.

32. It will be observed that in the previous example the same point P_2 is denoted by each of the four sets of polar coordinates

$$(3,\ 150°),\ (-3,\ 330°),\ (3,\ -210°) \text{ and } (-3,\ -30°).$$

In general it will be found that the same point is given by each of the polar coordinates

$$(r,\ \theta),\ (-r,\ 180°+\theta),\ \{r,\ -(360°-\theta)\} \text{ and } \{-r,\ -(180°-\theta)\},$$

or, expressing the angles in radians, by each of the coordinates

$$(r,\ \theta),\ (-r,\ \pi+\theta),\ \{r,\ -(2\pi-\theta)\} \text{ and } \{-r,\ -(\pi-\theta)\}.$$

It is also clear that adding 360° (or any multiple of 360°) to the vectorial angle does not alter the final position of the revolving line, so that $(r,\ \theta)$ is always the same point as $(r,\ \theta+n\,.\,360°)$, where n is an integer.

So, adding 180° or any odd multiple of 180° to the vectorial angle and changing the sign of the radius vector gives the same point as before. Thus the point

$$[-r,\ \theta+(2n+1)\,180°]$$

is the same point as $[-r,\ \theta+180°]$, *i.e.* is the point $[r,\ \theta]$.

33. *To find the length of the straight line joining two points whose polar coordinates are given.*

Let A and B be the two points and let their polar coordinates be $(r_1,\ \theta_1)$ and $(r_2,\ \theta_2)$ respectively, so that

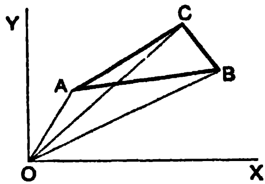
$$OA=r_1,\ OB=r_2,\ \angle XOA=\theta_1, \text{ and } \angle XOB=\theta_2.$$

Then (*Trigonometry*, Art. 164)

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos AOB$$
$$= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2).$$

34. *To find the area of a triangle the coordinates of whose angular points are given.*

Let ABC be the triangle and let (r_1, θ_1), (r_2, θ_2), and (r_3, θ_3) be the polar coordinates of its angular points.



We have

$$\triangle ABC = \triangle OBC + \triangle OCA - \triangle OBA \quad \ldots\ldots(1).$$

Now

$$\triangle OBC = \tfrac{1}{2} OB \cdot OC \sin BOC$$
[*Trigonometry*, Art. 198]
$$= \tfrac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2).$$

So $\triangle OCA = \frac{1}{2} OC \cdot OA \sin COA = \frac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3)$,

and $\triangle OAB = \frac{1}{2} OA \cdot OB \sin AOB = \frac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2)$

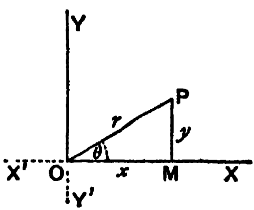
$$= -\tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1).$$

Hence (1) gives

$$\triangle ABC = \tfrac{1}{2}\left[r_2r_3 \sin(\theta_3 - \theta_2) + r_3r_1 \sin(\theta_1 - \theta_3) + r_1r_2 \sin(\theta_2 - \theta_1)\right].$$

35. *To change from Cartesian Coordinates to Polar Coordinates, and conversely.*

Let P be any point whose Cartesian coordinates, referred to rectangular axes, are x and y, and whose polar coordinates, referred to O as pole and OX as initial line, are (r, θ).



Draw PM perpendicular to OX so that we have

$OM = x$, $MP = y$, $\angle MOP = \theta$,

and $OP = r$.

From the triangle MOP we have

$$x = OM = OP \cos MOP = r\cos\theta \quad \ldots\ldots\ldots(1),$$
$$y = MP = OP \sin MOP = r\sin\theta \quad \ldots\ldots\ldots(2),$$
$$r = OP = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2} \quad \ldots\ldots(3),$$

and

$$\tan\theta = \frac{MP}{OM} = \frac{y}{x} \text{ ....(4).}$$

Equations (1) and (2) express the Cartesian coordinates in terms of the polar coordinates.

Equations (3) and (4) express the polar in terms of the Cartesian coordinates.

The same relations will be found to hold if P be in any other of the quadrants into which the plane is divided by XOX' and YOY'.

Ex. *Change to Cartesian coordinates the equations*

$$(1)\ r = a\sin\theta, \text{ and } (2)\ r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}.$$

(1) Multiplying the equation by r, it becomes $r^2 = ar\sin\theta$, *i.e.* by equations (2) and (3), $x^2+y^2=ay$.

(2) Squaring the equation (2), it becomes

$$r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta),$$

i.e. $$2r^2 = ar + ar\cos\theta,$$

i.e. $$2(x^2+y^2) = a\sqrt{x^2+y^2} + ax,$$

i.e. $$(2x^2+2y^2-ax)^2 = a^2(x^2+y^2).$$

EXAMPLES. III.

Lay down the positions of the points whose polar coordinates are

1. $(3, 45°)$. 2. $(-2, -60°)$. 3. $(4, 135°)$. 4. $(2, 330°)$.

5. $(-1, -180°)$. 6. $(1, -210°)$. 7. $(5, -675°)$. 8. $\left(a, \frac{\pi}{2}\right)$.

9. $\left(2a, -\frac{\pi}{2}\right)$. 10. $\left(-a, \frac{\pi}{6}\right)$. 11. $\left(-2a, -\frac{2\pi}{3}\right)$.

Find the lengths of the straight lines joining the pairs of points whose polar coordinates are

12. $(2, 30°)$ and $(4, 120°)$. 13. $(-3, 45°)$ and $(7, 105°)$.

14. $\left(a, \frac{\pi}{2}\right)$ and $\left(3a, \frac{\pi}{6}\right)$.

15. Prove that the points (0, 0), $\left(3, \frac{\pi}{2}\right)$, and $\left(3, \frac{\pi}{6}\right)$ form an equilateral triangle.

Find the areas of the triangles the coordinates of whose angular points are

16. (1, 30°), (2, 60°), and (3, 90°).

17. (−3, −30°), (5, 150°), and (7, 210°).

18. $\left(-a, \frac{\pi}{6}\right)$, $\left(a, \frac{\pi}{2}\right)$, and $\left(-2a, -\frac{2\pi}{3}\right)$.

Find the polar coordinates (drawing the figure in each case) of the points

19. $x=\sqrt{3},\ y=1$. 20. $x=-\sqrt{3},\ y=1$. 21. $x=-1,\ y=1$.

Find the Cartesian coordinates (drawing a figure in each case) of the points whose polar coordinates are

22. $\left(5, \frac{\pi}{4}\right)$. 23. $\left(-5, \frac{\pi}{3}\right)$. 24. $\left(5, -\frac{\pi}{4}\right)$.

Change to polar coordinates the equations

25. $x^2+y^2=a^2$. 26. $y=x\tan\alpha$. 27. $x^2+y^2=2ax$.

28. $x^2-y^2=2ay$. 29. $x^3=y^2(2a-x)$. 30. $(x^2+y^2)^2=a^2(x^2-y^2)$.

Transform to Cartesian coordinates the equations

31. $r=a$. 32. $\theta=\tan^{-1}m$. 33. $r=a\cos\theta$.

34. $r=a\sin 2\theta$. 35. $r^2=a^2\cos 2\theta$. 36. $r^2\sin 2\theta=2a^2$.

37. $r^2\cos 2\theta=a^2$. 38. $r^{\frac{1}{2}}\cos\frac{\theta}{2}=a^{\frac{1}{2}}$. 39. $r^{\frac{1}{2}}=a^{\frac{1}{2}}\sin\frac{\theta}{2}$.

40. $r(\cos 3\theta+\sin 3\theta)=5k\sin\theta\cos\theta$.

CHAPTER III.

LOCUS. EQUATION TO A LOCUS.

36. WHEN a point moves so as always to satisfy a given condition, or conditions, the path it traces out is called its Locus under these conditions.

For example, suppose O to be a given point in the plane of the paper and that a point P is to move on the paper so that its distance from O shall be constant and equal to a. It is clear that all the positions of the moving point must lie on the circumference of a circle whose centre is O and whose radius is a. The circumference of this circle is therefore the "Locus" of P when it moves subject to the condition that its distance from O shall be equal to the constant distance a.

37. Again, suppose A and B to be two fixed points in the plane of the paper and that a point P is to move in the plane of the paper so that its distances from A and B are to be always equal. If we bisect AB in C and through it draw a straight line (of infinite length in both directions) perpendicular to AB, then any point on this straight line is at equal distances from A and B. Also there is no point, whose distances from A and B are the same, which does not lie on this straight line. This straight line is therefore the "Locus" of P subject to the assumed condition.

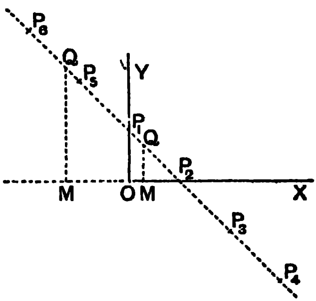
38. Again, suppose A and B to be two fixed points and that the point P is to move in the plane of the paper so that the angle APB is always a right angle. If we describe a circle on AB as diameter then P may be any

point on the circumference of this circle, since the angle in a semi-circle is a right angle; also it could easily be shewn that APB is not a right angle except when P lies on this circumference. The "Locus" of P under the assumed condition is therefore a circle on AB as diameter.

39. One single equation between two unknown quantities x and y, *e.g.*

$$x + y = 1 \quad \text{.......................(1)},$$

cannot completely determine the values of x and y.



Such an equation has an infinite number of solutions. Amongst them are the following:

$$\left.\begin{matrix} x=0,\\ y=1\end{matrix}\right\},\ \left.\begin{matrix} x=1,\\ y=0\end{matrix}\right\},\ \left.\begin{matrix} x=\ \ 2,\\ y=-1\end{matrix}\right\},\ \left.\begin{matrix} x=\ \ 3,\\ y=-2\end{matrix}\right\},\ \ldots$$

$$\left.\begin{matrix} x=-1,\\ y=\ \ 2\end{matrix}\right\},\ \left.\begin{matrix} x=-2,\\ y=\ \ 3\end{matrix}\right\},\ \ldots$$

Let us mark down on paper a number of points whose coordinates (as defined in the last chapter) satisfy equation (1).

Let OX and OY be the axes of coordinates.

If we mark off a distance $OP_1 (=1)$ along OY, we have a point P_1 whose coordinates (0, 1) clearly satisfy equation (1).

If we mark off a distance $OP_2 (=1)$ along OX, we have a point P_2 whose coordinates (1, 0) satisfy (1).

Similarly the point P_3, $(2, -1)$, and P_4, $(3, -2)$, satisfy the equation (1).

Again, the coordinates $(-1, 2)$ of P_5 and the coordinates $(-2, 3)$ of P_6 satisfy equation (1).

On making the measurements carefully we should find that all the points we obtain lie on the line P_1P_2 (produced both ways).

Again, if we took *any* point Q, lying on P_1P_2, and draw a perpendicular QM to OX, we should find on measurement that the sum of its x and y (each taken with its proper sign) would be equal to unity, so that the coordinates of Q would satisfy (1).

Also we should find no point, whose coordinates satisfy (1), which does not lie on P_1P_2.

All the points, lying on the straight line P_1P_2, and no others are therefore such that their coordinates satisfy the equation (1).

This result is expressed in the language of Analytical Geometry by saying that (1) is the Equation to the Straight Line P_1P_2.

40. Consider again the equation

$$x^2 + y^2 = 4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Amongst an infinite number of solutions of this equation are the following:

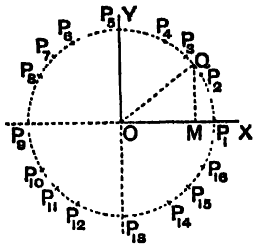
$$\left.\begin{matrix}x=2,\\ y=0\end{matrix}\right\}, \quad \left.\begin{matrix}x=\sqrt{3}\\ y=1\end{matrix}\right\}, \quad \left.\begin{matrix}x=\sqrt{2}\\ y=\sqrt{2}\end{matrix}\right\}, \quad \left.\begin{matrix}x=1\\ y=\sqrt{3}\end{matrix}\right\},$$

$$\left.\begin{matrix}x=0\\ y=2\end{matrix}\right\}, \quad \left.\begin{matrix}x=-1,\\ y=\sqrt{3}\end{matrix}\right\}, \quad \left.\begin{matrix}x=-\sqrt{2},\\ y=\sqrt{2}\end{matrix}\right\}, \quad \left.\begin{matrix}x=-\sqrt{3},\\ y=1\end{matrix}\right\},$$

$$\left.\begin{matrix}x=-2,\\ y=0\end{matrix}\right\}, \quad \left.\begin{matrix}x=-\sqrt{3},\\ y=-1\end{matrix}\right\}, \quad \left.\begin{matrix}x=-\sqrt{2},\\ y=-\sqrt{2}\end{matrix}\right\}, \quad \left.\begin{matrix}x=-1,\\ y=-\sqrt{3}\end{matrix}\right\},$$

$$\left.\begin{matrix}x=0,\\ y=-2\end{matrix}\right\}, \quad \left.\begin{matrix}x=1,\\ y=-\sqrt{3}\end{matrix}\right\}, \quad \left.\begin{matrix}x=\sqrt{2},\\ y=-\sqrt{2}\end{matrix}\right\}, \text{ and } \left.\begin{matrix}x=\sqrt{3}\\ y=-1\end{matrix}\right\}.$$

All these points are respectively represented by the points $P_1, P_2, P_3, \ldots P_{16}$, and they will all be found to lie on the dotted circle whose centre is O and radius is 2.



Also, if we take any other point Q on this circle and its ordinate QM, it follows, since $OM^2 + MQ^2 = OQ^2 = 4$, that the x and y of the point Q satisfies (1).

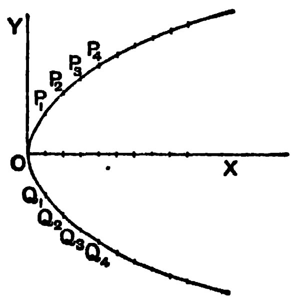
The dotted circle therefore passes through all the points whose coordinates satisfy (1).

In the language of Analytical Geometry the equation (1) is therefore the equation to the above circle.

41. As another example let us trace the locus of the point whose coordinates satisfy the equation

$$y^2 = 4x \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If we give x a negative value we see that y is impossible; for the square of a real quantity cannot be negative.



We see therefore that there are no points lying to the left of OY.

If we give x any positive value we see that y has two real corresponding values which are equal and of opposite signs.

The following values, amongst an infinite number of others, satisfy (1), viz.

$$\left.\begin{matrix} x=0, \\ y=0 \end{matrix}\right\}, \quad \left.\begin{matrix} x=1, \\ y=+2 \text{ or } -2 \end{matrix}\right\}, \quad \left.\begin{matrix} x=2, \\ y=2\sqrt{2} \text{ or } -2\sqrt{2} \end{matrix}\right\},$$

$$\left.\begin{matrix} x=4 \\ y=+4 \text{ or } -4 \end{matrix}\right\}, \ldots \left.\begin{matrix} x=16, \\ y=8 \text{ or } -8 \end{matrix}\right\}, \ldots \left.\begin{matrix} x=+\infty, \\ y \quad +\infty \text{ or } -\infty \end{matrix}\right\}.$$

The origin is the first of these points and P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, … represent the next pairs of points.

If we took a large number of values of x and the corresponding values of y, the points thus obtained would be found all to lie on the curve in the figure.

Both of its branches would be found to stretch away to infinity towards the right of the figure.

Also, if we took any point on this curve and measured with sufficient accuracy its x and y the values thus obtained would be found to satisfy equation (1).

Also we should not be able to find any point, not lying on the curve, whose coordinates would satisfy (1).

In the language of Analytical Geometry the equation (1) is the equation to the above curve. This curve is called a Parabola and will be fully discussed in Chapter X.

42. If a point move so as to satisfy any given condition it will describe some definite curve, or locus, and there can always be found an equation between the x and y of *any* point on the path.

This equation is called the equation to the locus or curve. Hence

Def. Equation to a curve. *The equation to a curve is the relation which exists between the coordinates of* **any** *point on the curve, and which holds for no other points except those lying on the curve.*

43. Conversely to every equation between x and y it will be found that there is, in general, a definite geometrical locus.

Thus in Art. 39 the equation is $x+y=1$, and the definite path, or locus, is the straight line P_1P_2 (produced indefinitely both ways).

In Art. 40 the equation is $x^2+y^2=4$, and the definite path, or locus, is the dotted circle.

Again the equation $y=1$ states that the moving point is such that its ordinate is always unity, *i.e.* that it is always at a distance 1 from the axis of x. The definite path, or locus, is therefore a straight line parallel to OX and at a distance unity from it.

44. In the next chapter it will be found that if the equation be of the first degree (*i.e.* if it contain no products, squares, or higher powers of x and y) the locus corresponding is always a straight line.

If the equation be of the second or higher degree, the corresponding locus is, in general, a curved line.

45. We append a few simple examples of the formation of the equation to a locus.

Ex. 1. *A point moves so that the algebraic sum of its distances from two given perpendicular axes is equal to a constant quantity a; find the equation to its locus.*

Take the two straight lines as the axes of coordinates. Let (x, y) be any point satisfying the given condition. We then have $x+y=a$.

This being the relation connecting the coordinates of any point on the locus is the equation to the locus.

It will be found in the next chapter that this equation represents a straight line.

Ex. 2. *The sum of the squares of the distances of a moving point from the two fixed points* $(a, 0)$ *and* $(-a, 0)$ *is equal to a constant quantity* $2c^2$. *Find the equation to its locus.*

Let (x, y) be any position of the moving point. Then, by Art. 20, the condition of the question gives

$$\{(x-a)^2+y^2\}+\{(x+a)^2+y^2\}=2c^2,$$

i.e.
$$x^2+y^2=c^2-a^2.$$

This being the relation between the coordinates of any, and every, point that satisfies the given condition is, by Art. 42, the equation to the required locus.

This equation tells us that the square of the distance of the point (x, y) from the origin is constant and equal to c^2-a^2, and therefore the locus of the point is a circle whose centre is the origin.

Ex. 3. *A point moves so that its distance from the point* $(-1, 0)$ *is always three times its distance from the point* $(0, 2)$.

Let (x, y) be *any* point which satisfies the given condition. We then have

$$\sqrt{(x+1)^2+(y-0)^2}=3\sqrt{(x-0)^2+(y-2)^2},$$

so that, on squaring,

$$x^2+2x+1+y^2=9(x^2+y^2-4y+4),$$

i.e.
$$8(x^2+y^2)-2x-36y+35=0.$$

This being the relation between the coordinates of each, and every, point that satisfies the given relation is, by Art. 42, the required equation.

It will be found, in a later chapter, that this equation represents a circle.

EXAMPLES. IV.

By taking a number of solutions, as in Arts. 39—41, sketch the loci of the following equations:

1. $2x+3y=10$. 2. $4x-y=7$. 3. $x^2-2ax+y^2=0$.
4. $x^2-4ax+y^2+3a^2=0$. 5. $y^2=x$. 6. $3x=y^2-9$.
7. $\frac{x^2}{4}+\frac{y^2}{9}=1$.

A and B being the fixed points $(a, 0)$ and $(-a, 0)$ respectively, obtain the equations giving the locus of P, when

8. $PA^2-PB^2=$ a constant quantity $=2k^2$.

9. $PA=nPB$, n being constant.

10. $PA+PB=c$, a constant quantity.

11. $PB^2+PC^2=2PA^2$, C being the point $(c, 0)$.

12. Find the locus of a point whose distance from the point (1, 2) is equal to its distance from the axis of y.

Find the equation to the locus of a point which is always equidistant from the points whose coordinates are

13. (1, 0) and (0, −2). 14. (2, 3) and (4, 5).

15. $(a+b, a-b)$ and $(a-b, a+b)$.

Find the equation to the locus of a point which moves so that

16. its distance from the axis of x is three times its distance from the axis of y.

17. its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

18. the sum of the squares of its distances from the axes is equal to 3.

19. the square of its distance from the point (0, 2) is equal to 4.

20. its distance from the point (3, 0) is three times its distance from (0, 2).

21. its distance from the axis of x is always one half its distance from the origin.

22. A fixed point is at a perpendicular distance a from a fixed straight line and a point moves so that its distance from the fixed point is always equal to its distance from the fixed line. Find the equation to its locus, the axes of coordinates being drawn through the fixed point and being parallel and perpendicular to the given line.

23. In the previous question if the first distance be (1), always half, and (2), always twice, the second distance, find the equations to the respective loci.

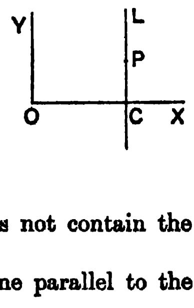
CHAPTER IV.

THE STRAIGHT LINE. RECTANGULAR COORDINATES.

46. *To find the equation to a straight line which is parallel to one of the coordinate axes.*

Let CL be any line parallel to the axis of y and passing through a point C on the axis of x such that $OC = c$.

Let P be any point on this line whose coordinates are x and y.



Then the abscissa of the point P is always c, so that

$$x = c \qquad \text{..........(1).}$$

This being true for every point on the line CL (produced indefinitely both ways), and for no other point, is, by Art. 42, the equation to the line.

It will be noted that the equation does not contain the coordinate y.

Similarly the equation to a straight line parallel to the axis of x is $y = d$.

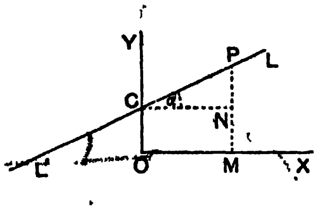
Cor. The equation to the axis of x is $y = 0$.

The equation to the axis of y is $x = 0$.

47. *To find the equation to a straight line which cuts off a given intercept on the axis of y and is inclined at a given angle to the axis of x.*

Let the given intercept be c and let the given angle be α.

Let C be a point on the axis of y such that OC is c. Through C draw a straight line LCL' inclined at an angle α $(=\tan^{-1} m)$ to the axis of x, so that $\tan\alpha = m$.



The straight line LCL' is therefore the straight line required, and we have to find the relation between the coordinates of any point P lying on it.

Draw PM perpendicular to OX to meet in N a line through C parallel to OX.

Let the coordinates of P be x and y, so that $OM = x$ and $MP = y$.

Then $MP = NP + MN = CN \tan\alpha + OC = m \cdot x + c$,

i.e. $$\mathbf{y = mx + c.}$$

This relation being true for *any* point on the given straight line is, by Art. 42, the equation to the straight line.

[In this, and other similar cases, it could be shewn, conversely, that the equation is only true for points lying on the given straight line.]

Cor. The equation to any straight line passing through the origin, *i.e.* which cuts off a zero intercept from the axis of y, is found by putting $c = 0$ and hence is $y = mx$.

48. The angle α which is used in the previous article is the angle through which a straight line, originally parallel to OX, would have to turn in order to coincide with the given direction, the rotation being always in the positive direction. Also m is always the tangent of this angle. In the case of such a straight line as AB, in the figure of Art. 50, m is equal to the tangent of the angle XAP (not of the angle PAO). In this case therefore m, being the tangent of an obtuse angle, is a negative quantity.

The student should verify the truth of the equation of the last article for *all* points on the straight line LCL', and also for straight lines in other positions, *e.g.* for such a straight line as A_2B_2 in the figure of Art. 59. In this latter case both m and c are negative quantities.

A careful consideration of all the possible cases of a few propositions will soon satisfy him that this verification is not always necessary, but that it is sufficient to consider the standard figure.

49. Ex. The equation to the straight line cutting off an intercept 3 from the negative direction of the axis of y, and inclined at 120° to the axis of x, is

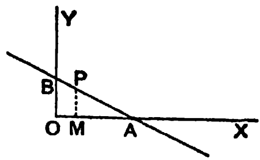
$$y = x \tan 120° + (-3),$$

i.e. $$y = -x\sqrt{3} - 3,$$

i.e. $$y + x\sqrt{3} + 3 = 0.$$

50. *To find the equation to the straight line which cuts off given intercepts a and b from the axes.*

Let A and B be on OX and OY respectively, and be such that $OA = a$ and $OB = b$.



Join AB and produce it indefinitely both ways. Let P be any point (x, y) on this straight line, and draw PM perpendicular to OX.

We require the relation that always holds between x and y, so long as P lies on AB.

By geometry, we have

$$\frac{OM}{OA} = \frac{PB}{AB}, \text{ and } \frac{MP}{OB} = \frac{AP}{AB}.$$

$$\therefore \frac{OM}{OA} + \frac{MP}{OB} = \frac{PB + AP}{AB} = 1,$$

i.e. $$\frac{x}{a} + \frac{y}{b} = 1.$$

This is therefore the required equation; for it is the relation that holds between the coordinates of *any* point lying on the given straight line.

51. The equation in the preceding article may be also obtained by expressing the fact that the sum of the areas of the triangles OPA and OPB is equal to OAB, so that

$$\tfrac{1}{2}a \times y + \tfrac{1}{2}b \times x = \tfrac{1}{2}a \times b,$$

and hence $$\frac{x}{a} + \frac{y}{b} = 1.$$

52. Ex. 1. *Find the equation to the straight line passing through the point* (3, −4) *and cutting off intercepts, equal but of opposite signs, from the two axes.*

Let the intercepts cut off from the two axes be of lengths a and $-a$.

The equation to the straight line is then

$$\frac{x}{a}+\frac{y}{-a}=1,$$

i.e. $$x-y=a \quad \text{.................................(1).}$$

Since, in addition, the straight line is to go through the point $(3, -4)$, these coordinates must satisfy (1), so that

$$3-(-4)=a,$$

and therefore $$a=7.$$

The required equation is therefore

$$x-y=7.$$

Ex. 2. *Find the equation to the straight line which passes through the point $(-5, 4)$ and is such that the portion of it between the axes is divided by the point in the ratio of* $1:2$.

Let the required straight line be $\frac{x}{a}+\frac{y}{b}=1$. This meets the axes in the points whose coordinates are $(a, 0)$ and $(0, b)$.

The coordinates of the point dividing the line joining these points in the ratio $1:2$, are (Art. 22)

$$\frac{2.a+1.0}{2+1} \text{ and } \frac{2.0+1.b}{2+1}, \text{ i.e. } \frac{2a}{3} \text{ and } \frac{b}{3}.$$

If this be the point $(-5, 4)$ we have

$$-5=\frac{2a}{3} \text{ and } 4=\frac{b}{3},$$

so that $$a=-\tfrac{15}{2} \text{ and } b=12.$$

The required straight line is therefore

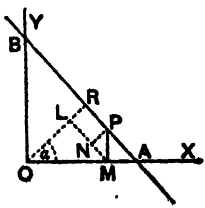
$$\frac{x}{-\frac{15}{2}}+\frac{y}{12}=1,$$

i.e. $$5y-8x=60.$$

53. *To find the equation to a straight line in terms of the perpendicular let fall upon it from the origin and the angle that this perpendicular makes with the axis of x.*

Let OR be the perpendicular from O and let its length be p.

Let α be the angle that OR makes with OX.



Let P be any point, whose coordinates are x and y, lying on AB; draw the ordinate PM, and also ML perpendicular to OR and PN perpendicular to ML.

Then $$OL = OM \cos \alpha \quad \text{.......................(1)},$$

and $$LR = NP = MP \sin NMP.$$

But $$\angle NMP = 90^\circ - \angle NMO = \angle MOL = \alpha.$$

$$\therefore \quad LR = MP \sin \alpha \quad \text{....................(2)}.$$

Hence, adding (1) and (2), we have

$$OM \cos \alpha + MP \sin \alpha = OL + LR = OR = p,$$

i.e. $$\mathbf{x \cos \alpha + y \sin \alpha = p.}$$

This is the required equation.

54. In Arts. 47—53 we have found that the corresponding equations are only of the first degree in x and y. We shall now prove that

Any equation of the first degree in x and y always represents a straight line.

For the most general form of such an equation is

$$Ax + By + C = 0 \quad \text{.................(1)},$$

where A, B, and C are constants, *i.e.* quantities which do not contain x and y and which remain the same for all points on the locus.

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be *any* three points on the locus of the equation (1).

Since the point (x_1, y_1) lies on the locus, its coordinates when substituted for x and y in (1) must satisfy it.

Hence $$Ax_1 + By_1 + C = 0 \quad \text{.................(2)}.$$

So $$Ax_2 + By_2 + C = 0 \quad \text{.................(3)},$$

and $$Ax_3 + By_3 + C = 0 \quad \text{.................(4)}.$$

Since these three equations hold between the three quantities A, B, and C, we can, as in Art. 12, eliminate them.

The result is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0 \quad \text{.................(5)}.$$

But, by Art. 25, the relation (5) states that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is zero.

Also these are any three points on the locus.

The locus must therefore be a straight line; for a curved line could not be such that the triangle obtained by joining *any* three points on it should be zero.

55. The proposition of the preceding article may also be deduced from Art. 47. For the equation

$$Ax+By+C=0$$

may be written

$$y=-\frac{A}{B}x-\frac{C}{B},$$

and this is the same as the straight line

$$y=mx+c,$$

if

$$m=-\frac{A}{B} \text{ and } c=-\frac{C}{B}.$$

But in Art. 47 it was shewn that $y=mx+c$ was the equation to a straight line cutting off an intercept c from the axis of y and inclined at an angle $\tan^{-1}m$ to the axis of x.

The equation $$Ax+By+C=0$$

therefore represents a straight line cutting off an intercept $-\frac{C}{B}$ from the axis of y and inclined at an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ to the axis of x.

56. We can reduce the general equation of the first degree

$$Ax+By+C=0 \text{ ....(1)}$$

to the form of Art. 53.

For, if p be the perpendicular from the origin on (1) and a the angle it makes with the axis, the equation to the straight line must be

$$x\cos a+y\sin a-p=0 \text{ ....(2).}$$

This equation must therefore be the same as (1).

Hence $$\frac{\cos a}{A}=\frac{\sin a}{B}=\frac{-p}{C},$$

i.e. $$\frac{p}{C}=\frac{\cos a}{-A}=\frac{\sin a}{-B}=\frac{\sqrt{\cos^2 a+\sin^2 a}}{\sqrt{A^2+B^2}}=\frac{1}{\sqrt{A^2+B^2}}.$$

Hence

$$\cos a=\frac{-A}{\sqrt{A^2+B^2}},\quad \sin a=\frac{-B}{\sqrt{A^2+B^2}},\quad \text{and } p=\frac{C}{\sqrt{A^2+B^2}}.$$

The equation (1) may therefore be reduced to the form (2) by dividing it by $\sqrt{A^2+B^2}$ and arranging it so that the constant term is negative.

57. Ex. *Reduce to the perpendicular form the equation*

$$x+y\sqrt{3}+7=0 \qquad \text{.............................(1).}$$

Here $\qquad \sqrt{A^2+B^2}=\sqrt{1+3}=\sqrt{4}=2.$

Dividing (1) by 2, we have

$$\tfrac{1}{2}x+y\frac{\sqrt{3}}{2}+\tfrac{7}{2}=0,$$

i.e. $$x\left(-\tfrac{1}{2}\right)+y\left(-\frac{\sqrt{3}}{2}\right)-\tfrac{7}{2}=0,$$

i.e. $$x\cos 240^\circ+y\sin 240^\circ-\tfrac{7}{2}=0.$$

58. *To trace the straight line given by an equation of the first degree.*

Let the equation be

$$Ax+By+C=0 \qquad \text{....................(1).}$$

(α) This can be written in the form

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1.$$

Comparing this with the result of Art. 50, we see that it represents a straight line which cuts off intercepts $-\frac{C}{A}$ and $-\frac{C}{B}$ from the axes. Its position is therefore known.

If C be zero, the equation (1) reduces to the form

$$y=-\frac{A}{B}x,$$

and thus (by Art. 47, Cor.) represents a straight line passing through the origin inclined at an angle $\tan^{-1}\left(-\frac{A}{B}\right)$ to the axis of x. Its position is therefore known.

(β) The straight line may also be traced by finding the coordinates of any two points on it.

If we put $y=0$ in (1) we have $x=-\frac{C}{A}$. The point $\left(-\frac{C}{A},\ 0\right)$ therefore lies on it.

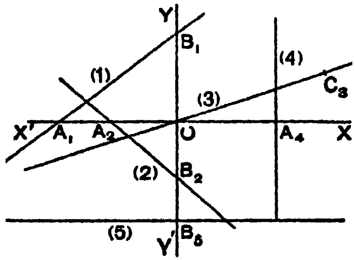
If we put $x=0$, we have $y=-\frac{C}{B}$, so that the point $\left(0, -\frac{C}{B}\right)$ lies on it.

Hence, as before, we have the position of the straight line.

59. Ex. *Trace the straight lines*

(1) $3x-4y+7=0$; (2) $7x+8y+9=0$;

(3) $3y=x$; (4) $x=2$; (5) $y=-2$.



(1) Putting $y=0$, we have $x=-\frac{7}{3}$,
and putting $x=0$, we have $y=\frac{7}{4}$.

Measuring $OA_1 \left(=-\frac{7}{3}\right)$ along the axis of x we have one point on the line.

Measuring $OB_1 \left(=\frac{7}{4}\right)$ along the axis of y we have another point.

Hence A_1B_1, produced both ways, is the required line.

(2) Putting in succession y and x equal to zero, we have the intercepts on the axes equal to $-\frac{9}{7}$ and $-\frac{9}{8}$.

If then $OA_2=-\frac{9}{7}$ and $OB_2=-\frac{9}{8}$, we have A_2B_2 the required line.

(3) The point (0, 0) satisfies the equation so that the origin is on the line.

Also the point (3, 1), *i.e.* C_3, lies on it. The required line is therefore OC_3.

(4) The line $x=2$ is, by Art. 46, parallel to the axis of y and passes through the point A_4 on the axis of x such that $OA_4=2$.

(5) The line $y=-2$ is parallel to the axis of x and passes through the point B_5 on the axis of y, such that $OB_5=-2$.

60. Straight Line at Infinity. We have seen that the equation $Ax+By+C=0$ represents a straight line

which cuts off intercepts $-\frac{C}{A}$ and $-\frac{C}{B}$ from the axes of coordinates.

If A vanish, but not B or C, the intercept on the axis of x is infinitely great. The equation of the straight line then reduces to the form $y=$ constant, and hence, as in Art. 46, represents a straight line parallel to Ox.

So if B vanish, but not A or C, the straight line meets the axis of y at an infinite distance and is therefore parallel to it.

If A and B both vanish, but not C, these two intercepts are both infinite and therefore the straight line $0 \cdot x + 0 \cdot y + C = 0$ is altogether at infinity.

61. The multiplication of an equation by a constant does not alter it. Thus the equations

$$2x - 3y + 5 = 0 \text{ and } 10x - 15y + 25 = 0$$

represent the same straight line.

Conversely, if two equations of the first degree represent the same straight line, one equation must be equal to the other multiplied by a constant quantity, so that the ratios of the corresponding coefficients must be the same.

For example, if the equations

$$a_1x + b_1y + c_1 = 0 \text{ and } A_1x + B_1y + C_1 = 0$$

represent the same straight line, we must have

$$\frac{a_1}{A_1} = \frac{b_1}{B_1} = \frac{c_1}{C_1}.$$

62. *To find the equation to the straight line which passes through the two given points (x', y') and (x'', y'').*

By Art. 47, the equation to **any** straight line is

$$y = mx + c \text{.......................(1).}$$

By properly determining the quantities m and c we can make (1) represent any straight line we please.

If (1) pass through the point (x', y'), we have

$$y' = mx' + c \text{.......................(2).}$$

Substituting for c from (2), the equation (1) becomes

$$\mathbf{y - y' = m(x - x')} \text{.................(3).}$$

This is the equation to the line going through (x', y') making an angle $\tan^{-1} m$ with OX. If in addition (3) passes through the point (x'', y''), then

$$y'' - y' = m(x'' - x'),$$

giving
$$m = \frac{y'' - y'}{x'' - x'}.$$

Substituting this value in (3), we get as the required equation

$$\mathbf{y - y' = \frac{y'' - y'}{x'' - x'}(x - x').}$$

63. Ex. *Find the equation to the straight line which passes through the points* $(-1, 3)$ *and* $(4, -2)$.

Let the required equation be

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Since (1) goes through the first point, we have

$$3 = -m + c, \text{ so that } c = m + 3.$$

Hence (1) becomes

$$y = mx + m + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If in addition the line goes through the second point, we have

$$-2 = 4m + m + 3, \text{ so that } m = -1.$$

Hence (2) becomes

$$y = -x + 2, \text{ i.e. } x + y = 2.$$

Or, again, using the result of the last article the equation is

$$y - 3 = \frac{-2-3}{4-(-1)}(x+1) = -x - 1,$$

i.e.
$$y + x = 2.$$

64. To fix definitely the position of a straight line we must have always two quantities given. Thus one point on the straight line and the direction of the straight line will determine it; or again two points lying on the straight line will determine it.

Analytically, the general equation to a straight line will contain two arbitrary constants, which will have to be determined so that the general equation may represent any particular straight line.

Thus, in Art. 47, the quantities m and c which remain the same, *so long as we are considering the same straight line*, are the two constants for the straight line.

Similarly, in Art. 50, the quantities a and b are the constants for the straight line.

65. In any equation to a locus the quantities x and y, which are the coordinates of any point on the locus, are called Current Coordinates; the curve may be conceived as traced out by a point which "runs" along the locus.

EXAMPLES. V.

Find the equation to the straight line

1. cutting off an intercept unity from the positive direction of the axis of y and inclined at $45°$ to the axis of x.

2. cutting off an intercept -5 from the axis of y and being equally inclined to the axes.

3. cutting off an intercept 2 from the negative direction of the axis of y and inclined at $30°$ to OX.

4. cutting off an intercept -3 from the axis of y and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the axis of x.

Find the equation to the straight line

5. cutting off intercepts 3 and 2 from the axes.

6. cutting off intercepts -5 and 6 from the axes.

7. Find the equation to the straight line which passes through the point $(5, 6)$ and has intercepts on the axes

   (1) equal in magnitude and both positive,

   (2) equal in magnitude but opposite in sign.

8. Find the equations to the straight lines which pass through the point $(1, -2)$ and cut off equal distances from the two axes.

9. Find the equation to the straight line which passes through the given point (x', y') and is such that the given point bisects the part intercepted between the axes.

10. Find the equation to the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio $5 : 3$.

Trace the straight lines whose equations are

11. $x+2y+3=0$. 12. $5x-7y-9=0$.

13. $3x+7y=0$. 14. $2x-3y+4=0$.

Find the equations to the straight lines passing through the following pairs of points.

15. $(0, 0)$ and $(2, -2)$. 16. $(3, 4)$ and $(5, 6)$.

17. $(-1, 3)$ and $(6, -7)$. 18. $(0, -a)$ and $(b, 0)$.

19. (a, b) and $(a+b, a-b)$.

20. $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. 21. $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2, \frac{a}{t_2}\right)$.

22. $(a \cos \phi_1, a \sin \phi_1)$ and $(a \cos \phi_2, a \sin \phi_2)$.

23. $(a \cos \phi_1, b \sin \phi_1)$ and $(a \cos \phi_2, b \sin \phi_2)$.

24. $(a \sec \phi_1, b \tan \phi_1)$ and $(a \sec \phi_2, b \tan \phi_2)$.

Find the equations to the sides of the triangles the coordinates of whose angular points are respectively

25. $(1, 4)$, $(2, -3)$, and $(-1, -2)$.

26. $(0, 1)$, $(2, 0)$, and $(-1, -2)$.

27. Find the equations to the diagonals of the rectangle the equations of whose sides are $x=a$, $x=a'$, $y=b$, and $y=b'$.

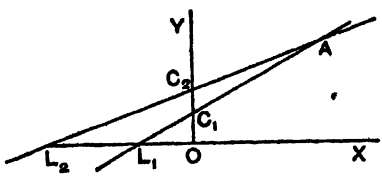
28. Find the equation to the straight line which bisects the distance between the points (a, b) and (a', b') and also bisects the distance between the points $(-a, b)$ and $(a', -b')$.

29. Find the equations to the straight lines which go through the origin and trisect the portion of the straight line $3x+y=12$ which is intercepted between the axes of coordinates.

Angles between straight lines.

66. *To find the angle between two given straight lines.*

Let the two straight lines be AL_1 and AL_2, meeting the axis of x in L_1 and L_2.



I. Let their equations be

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2 \quad \text{.........(1).}$$

By Art. 47 we therefore have

$$\tan AL_1X = m_1, \text{ and } \tan AL_2X = m_2.$$

Now $\qquad \angle L_1AL_2 = \angle AL_1X - \angle AL_2X.$

$$\tan L_1AL_2 = \tan [AL_1X - AL_2X]$$

$$= \frac{\tan AL_1X - \tan AL_2X}{1 + \tan AL_1X \cdot \tan AL_2X} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Hence the required angle $= \angle L_1AL_2$

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} \quad\text{.................. (2).}$$

[In any numerical example, if the quantity (2) be a positive quantity it is the tangent of the acute angle between the lines; if negative, it is the tangent of the obtuse angle.]

II. Let the equations of the straight lines be

$$A_1x + B_1y + C_1 = 0,$$

and
$$A_2x + B_2y + C_2 = 0.$$

By dividing the equations by B_1 and B_2, they may be written

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1},$$

and
$$y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2}.$$

Comparing these with the equations of (I.), we see that

$$m_1 = -\frac{A_1}{B_1}, \text{ and } m_2 = -\frac{A_2}{B_2}.$$

Hence the required angle

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} = \tan^{-1}\frac{-\frac{A_1}{B_1} - \left(-\frac{A_2}{B_2}\right)}{1 + \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)}$$

$$= \tan^{-1}\frac{B_1A_2 - A_1B_2}{A_1A_2 + B_1B_2} \quad\text{.................. (3).}$$

III. If the equations be given in the form

$$x\cos\alpha + y\sin\alpha - p_1 = 0 \quad\text{and}\quad x\cos\beta + y\sin\beta - p_2 = 0,$$

the perpendiculars from the origin make angles α and β with the axis of x.

Now that angle between two straight lines, in which the origin lies, is the supplement of the angle between the perpendiculars, and the angle between these perpendiculars is $\beta - \alpha$.

[For, if OR_1 and OR_2 be the perpendiculars from the origin upon the two lines, then the points O, R_1, R_2, and A lie on a circle, and hence the angles R_1OR_2 and R_2AR_1 are either equal or supplementary.]

67. *To find the condition that two straight lines may be parallel.*

Two straight lines are parallel when the angle between them is zero and therefore the tangent of this angle is zero.

The equation (2) of the last article then gives

$$\mathbf{m_1 = m_2}.$$

Two straight lines whose equations are given in the "m" form are therefore parallel when their "m's" are the same, or, in other words, if their equations differ only in the constant term.

The straight line $Ax+By+C'=0$ is any straight line which is parallel to the straight line $Ax+By+C=0$. For the "m's" of the two equations are the same.

Again the equation $A(x-x')+B(y-y')=0$ clearly represents the straight line which passes through the point (x', y') and is parallel to $Ax+By+C=0$.

The result (3) of the last article gives, as the condition for parallel lines,

$$B_1A_2 - A_1B_2 = 0,$$

i.e.
$$\frac{A_1}{B_1} = \frac{A_2}{B_2}.$$

68. Ex. *Find the equation to the straight line, which passes through the point* $(4, -5)$, *and which is parallel to the straight line*

$$3x+4y+5=0\text{..............................(1)}.$$

Any straight line which is parallel to (1) has its equation of the form

$$3x+4y+C=0\text{..............................(2)}.$$

[For the "m" of both (1) and (2) is the same.]

This straight line will pass through the point $(4, -5)$ if

$$3\times 4+4\times(-5)+C=0,$$

i.e. if
$$C=20-12=8.$$

The equation (2) then becomes

$$3x+4y+8=0.$$

69. *To find the condition that two straight lines, whose equations are given, may be perpendicular.*

Let the straight lines be

$$y=m_1x+c_1,$$

and
$$y=m_2x+c_2.$$

If θ be the angle between them we have, by Art. 66,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \text{ .... } (1).$$

If the lines be perpendicular, then $\theta = 90°$, and therefore $\tan\theta = \infty$.

The right-hand member of equation (1) must therefore be infinite, and this can only happen when its denominator is zero.

The condition of perpendicularity is therefore that

$$1 + m_1 m_2 = 0, \quad i.e. \quad \mathbf{m_1 m_2 = -1.}$$

The straight line $y = m_2 x + c_2$ is therefore perpendicular to $y = m_1 x + c_1$, if $m_2 = -\frac{1}{m_1}$.

It follows that the straight lines

$$A_1 x + B_1 y + C_1 = 0 \quad \text{and} \quad A_2 x + B_2 y + C_2 = 0,$$

for which $m_1 = -\frac{A_1}{B_1}$ and $m_2 = -\frac{A_2}{B_2}$, are at right angles if

$$\left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right) = -1,$$

i.e. if
$$A_1 A_2 + B_1 B_2 = 0.$$

70. From the preceding article it follows that the two straight lines

$$A_1 x + B_1 y + C_1 = 0 \text{ .... } (1),$$

and
$$B_1 x - A_1 y + C_2 = 0 \text{ .... } (2),$$

are at right angles; for the product of their m's

$$= -\frac{A_1}{B_1} \times \frac{B_1}{A_1} = -1.$$

Also (2) is derived from (1) by interchanging the coefficients of x and y, changing the sign of one of them, and changing the constant into any other constant.

Ex. The straight line through (x', y') perpendicular to (1) is (2) where $B_1 x' - A_1 y' + C_2 = 0$, so that $C_2 = A_1 y' - B_1 x'$.

This straight line is therefore

$$B_1 (x - x') - A_1 (y - y') = 0$$

71. Ex. 1. *Find the equation to the straight line which passes through the point* $(4, -5)$ *and is perpendicular to the straight line*

$$3x+4y+5=0 \quad\text{..............................(1).}$$

First Method. Any straight line perpendicular to (1) is by the last article

$$4x-3y+C=0 \quad\text{..............................(2).}$$

[We should expect an arbitrary constant in (2) because there are an infinite number of straight lines perpendicular to (1).]

The straight line (2) passes through the point $(4, -5)$ if

$$4\times 4-3\times(-5)+C=0,$$

i.e. if $$C=-16-15=-31.$$

The required equation is therefore

$$4x-3y=31.$$

Second Method. Any straight line passing through the given point is

$$y-(-5)=m(x-4).$$

This straight line is perpendicular to (1) if the product of their m's is -1,

i.e. if $$m\times(-\tfrac{3}{4})=-1,$$

i.e. if $$m=\tfrac{4}{3}.$$

The required equation is therefore

$$y+5=\tfrac{4}{3}(x-4),$$

i.e. $$4x-3y=31.$$

Third Method. Any straight line is $y=mx+c$. It passes through the point $(4, -5)$, if

$$-5=4m+c \quad\text{..............................(3).}$$

It is perpendicular to (1) if

$$m\times(-\tfrac{3}{4})=-1 \quad\text{..............................(4).}$$

Hence $m=\frac{4}{3}$ and then (3) gives $c=-\frac{31}{3}$.

The required equation is therefore $y=\frac{4}{3}x-\frac{31}{3}$,

i.e. $$4x-3y=31.$$

[In the first method, we start with any straight line which is perpendicular to the given straight line and pick out that particular straight line which goes through the given point.

In the second method, we start with any straight line passing through the given point and pick out that particular one which is perpendicular to the given straight line.

In the third method, we start with any straight line whatever and determine its constants, so that it may satisfy the two given conditions.

The student should illustrate by figures.]

Ex. 2. *Find the equation to the straight line which passes through the point* (x', y') *and is perpendicular to the given straight line*

$$yy'=2a(x+x').$$

The given straight line is

$$yy' - 2ax - 2ax' = 0.$$

Any straight line perpendicular to it is (Art. 70)

$$2ay + xy' + C = 0 \quad\text{..........................} \quad (1).$$

This will pass through the point (x', y') and therefore will be the straight line required if the coordinates x' and y' satisfy it,

i.e. if $$2ay' + x'y' + C = 0,$$

i.e. if $$C = -2ay' - x'y'.$$

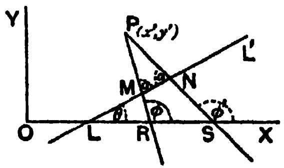
Substituting in (1) for C the required equation is therefore

$$2a(y - y') + y'(x - x') = 0.$$

72. *To find the equations to the straight lines which pass through a given point (x', y') and make a given angle α with the given straight line $y = mx + c$.*

Let P be the given point and let the given straight line be LMN, making an angle θ with the axis of x such that

$$\tan\theta = m.$$



In general (*i.e.* except when α is a right angle or zero) there are two straight lines PMR and PNS making an angle α with the given line.

Let these lines meet the axis of x in R and S and let them make angles ϕ and ϕ' with the positive direction of the axis of x.

The equations to the two required straight lines are therefore (by Art. 62)

$$y - y' = \tan\phi \times (x - x') \quad\text{.................}\quad (1),$$

and $$y - y' = \tan\phi' \times (x - x') \quad\text{.................}\quad (2).$$

Now $$\phi = \angle LMR + \angle RLM = \alpha + \theta,$$

and $$\phi' = \angle LNS + \angle SLN = (180^\circ - \alpha) + \theta.$$

Hence

$$\tan\phi = \tan(\alpha + \theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\tan\alpha + m}{1 - m\tan\alpha},$$

and $$\tan\phi' = \tan(180^\circ + \theta - \alpha)$$

$$= \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}.$$

On substituting these values in (1) and (2), we have as the required equations

$$y-y'=\frac{m+\tan\alpha}{1-m\tan\alpha}(x-x'),$$

and

$$y-y'=\frac{m-\tan\alpha}{1+m\tan\alpha}(x-x').$$

EXAMPLES. VI.

Find the angles between the pairs of straight lines

1. $x-y\sqrt{3}=5$ and $\sqrt{3}x+y=7$.

2. $x-4y=3$ and $6x-y=11$. 3. $y=3x+7$ and $3y-x=8$.

4. $y=(2-\sqrt{3})x+5$ and $y=(2+\sqrt{3})x-7$.

5. $(m^2-mn)y=(mn+n^2)x+n^3$ and $(mn+m^2)y=(mn-n^2)x+m^3$.

6. Find the tangent of the angle between the lines whose intercepts on the axes are respectively a, $-b$ and b, $-a$.

7. Prove that the points $(2, -1)$, $(0, 2)$, $(2, 3)$, and $(4, 0)$ are the coordinates of the angular points of a parallelogram and find the angle between its diagonals.

Find the equation to the straight line

8. passing through the point $(2, 3)$ and perpendicular to the straight line $4x-3y=10$.

9. passing through the point $(-6, 10)$ and perpendicular to the straight line $7x+8y=5$.

10. passing through the point $(2, -3)$ and perpendicular to the straight line joining the points $(5, 7)$ and $(-6, 3)$.

11. passing through the point $(-4, -3)$ and perpendicular to the straight line joining $(1, 3)$ and $(2, 7)$.

12. Find the equation to the straight line drawn at right angles to the straight line $\frac{x}{a}-\frac{y}{b}=1$ through the point where it meets the axis of x.

13. Find the equation to the straight line which bisects, and is perpendicular to, the straight line joining the points (a, b) and (a', b').

14. Prove that the equation to the straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and is perpendicular to the straight line $x\sec\theta+y\operatorname{cosec}\theta=a$ is $x\cos\theta-y\sin\theta=a\cos 2\theta$.

15. Find the equations to the straight lines passing through (x', y') and respectively perpendicular to the straight lines

$$xx'+yy'=a^2,$$

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

and $$x'y+xy'=a^2.$$

16. Find the equations to the straight lines which divide, internally and externally, the line joining $(-3, 7)$ to $(5, -4)$ in the ratio of $4:7$ and which are perpendicular to this line.

17. Through the point $(3, 4)$ are drawn two straight lines each inclined at $45°$ to the straight line $x-y=2$. Find their equations and find also the area included by the three lines.

18. Shew that the equations to the straight lines passing through the point $(3, -2)$ and inclined at $60°$ to the line

$\sqrt{3}x+y=1$ are $y+2=0$ and $y-\sqrt{3}x+2+3\sqrt{3}=0$.

19. Find the equations to the straight lines which pass through the origin and are inclined at $75°$ to the straight line

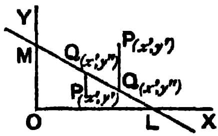
$$x+y+\sqrt{3}(y-x)=a.$$

20. Find the equations to the straight lines which pass through the point (h, k) and are inclined at an angle $\tan^{-1}m$ to the straight line $$y=mx+c.$$

21. Find the angle between the two straight lines $3x=4y+7$ and $5y=12x+6$ and also the equations to the two straight lines which pass through the point $(4, 5)$ and make equal angles with the two given lines.

73. *To shew that the point (x', y') is on one side or the other of the straight line $Ax+By+C=0$ according as the quantity $Ax'+By'+C$ is positive or negative.*

Let LM be the given straight line and P any point (x', y').



Through P draw PQ, parallel to the axis of y, to meet the given straight line in Q, and let the coordinates of Q be (x', y'').

Since Q lies on the given line, we have

$$Ax'+By''+C=0,$$

so that $$y''=-\frac{Ax'+C}{B}\text{ ....(1).}$$

It is clear from the figure that PQ is drawn parallel to the positive or negative direction of the axis of y according as P is on one side, or the other, of the straight line LM,

i.e. according as y'' is $>$ or $<$ y',

i.e. according as $y''-y'$ is positive or negative.

Now, by (1),

$$y'' - y' = -\frac{Ax' + C}{B} - y' = -\frac{1}{B}[Ax' + By' + C].$$

The point (x', y') is therefore on one side or the other of LM according as the quantity $Ax' + By' + C$ is negative or positive.

Cor. The point (x', y') and the origin are on the same side of the given line if $Ax' + By' + C$ and $A \times 0 + B \times 0 + C$ have the same signs, *i.e.* if $Ax' + By' + C$ has the same sign as C.

If these two quantities have opposite signs, then the origin and the point (x', y') are on opposite sides of the given line.

74. The condition that two points may lie on the same or opposite sides of a given line may also be obtained by considering the ratio in which the line joining the two points is cut by the given line.

For let the equation to the given line be

$$Ax + By + C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and let the coordinates of the two given points be (x_1, y_1) and (x_2, y_2).

The coordinates of the point which divides in the ratio $m_1 : m_2$ the line joining these points are, by Art. 22,

$$\frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \quad\ldots\ldots\ldots\ldots (2).$$

If this point lie on the given line we have

$$A\frac{m_1x_2 + m_2x_1}{m_1 + m_2} + B\frac{m_1y_2 + m_2y_1}{m_1 + m_2} + C = 0,$$

so that
$$\frac{m_1}{m_2} = -\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If the point (2) be *between* the two given points (x_1, y_1) and (x_2, y_2), *i.e.* if these two points be on *opposite* sides of the given line, the ratio $m_1 : m_2$ is positive.

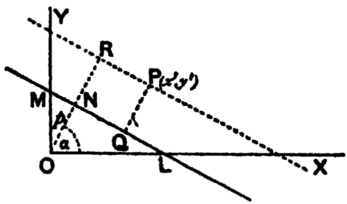
In this case, by (3) the two quantities $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs.

The two points (x_1, y_1) and (x_2, y_2) therefore lie on the op-

posite (or the same) sides of the straight line $Ax + By + C = 0$ according as the quantities $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite (or the same) signs.

Lengths of perpendiculars.

75. *To find the length of the perpendicular let fall from a given point upon a given straight line.*



(i) Let the equation of the straight line be

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

so that, if p be the perpendicular on it, we have

$$ON = p \text{ and } \angle XON = \alpha.$$

Let the given point P be (x', y').

Through P draw PR parallel to the given line to meet ON produced in R and draw PQ the required perpendicular.

If OR be p', the equation to PR is, by Art. 53,

$$x \cos \alpha + y \sin \alpha - p' = 0.$$

Since this passes through the point (x', y'), we have

$$x' \cos \alpha + y' \sin \alpha - p' = 0,$$

so that $$p' = x' \cos \alpha + y' \sin \alpha.$$

But the required perpendicular

$$= PQ = NR = OR - ON = p' - p$$

$$= \mathbf{x' \cos \alpha + y' \sin \alpha - p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The length of the required perpendicular is therefore obtained by substituting x' and y' for x and y in the given equation.

(ii) Let the equation to the straight line be

$$Ax + By + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the equation being written so that C is a negative quantity.

As in Art. 56 this equation is reduced to the form (1) by dividing it by $\sqrt{A^2+B^2}$. It then becomes

$$\frac{Ax}{\sqrt{A^2+B^2}}+\frac{By}{\sqrt{A^2+B^2}}+\frac{C}{\sqrt{A^2+B^2}}=0.$$

Hence

$$\cos\alpha=\frac{A}{\sqrt{A^2+B^2}},\quad \sin\alpha=\frac{B}{\sqrt{A^2+B^2}},\quad \text{and}\quad -p=\frac{C}{\sqrt{A^2+B^2}}.$$

The perpendicular from the point (x', y') therefore

$$=x'\cos\alpha+y'\sin\alpha-p$$

$$=\frac{\mathbf{Ax'+By'+C}}{\sqrt{\mathbf{A^2+B^2}}}.$$

The length of the perpendicular from (x', y') on (3) is therefore obtained by substituting x' and y' for x and y in the left-hand member of (3), and dividing the result so obtained by the square root of the sum of the squares of the coefficients of x and y.

Cor. 1. The perpendicular from the origin

$$=C\div\sqrt{A^2+B^2}.$$

Cor. 2. The length of the perpendicular is, by Art. 73, positive or negative according as (x', y') is on one side or the other of the given line.

76. The length of the perpendicular may also be obtained as follows:

As in the figure of the last article let the straight line meet the axes in L and M, so that

$$OL=-\frac{C}{A}\text{ and }OM=-\frac{C}{B}.$$

Let PQ be the perpendicular from P (x', y') on the given line and PS and PT the perpendiculars on the axes of coordinates.

We then have

$$\triangle PML+\triangle MOL=\triangle OLP+\triangle OPM,$$

i.e., since the area of a triangle is one half the product of its base and perpendicular height,

$$PQ.LM+OL.OM=OL.PS+OM.PT.$$

But $LM = \sqrt{\left(-\frac{C}{A}\right)^2 + \left(-\frac{C}{B}\right)^2} = \frac{\sqrt{A^2+B^2}}{AB} \times (-C)$,
since C is a negative quantity.

Hence

$$PQ \times \frac{\sqrt{A^2+B^2}}{AB} \times (-C) + \frac{C}{A} \cdot \frac{C}{B} = -\frac{C}{A} \times y' + \left(-\frac{C}{B}\right) \times x',$$

so that $$PQ = \frac{Ax' + By' + C}{\sqrt{A^2+B^2}}.$$

EXAMPLES. VII.

Find the length of the perpendicular drawn from

1. the point (4, 5) upon the straight line $3x+4y=10$.

2. the origin upon the straight line $\frac{x}{3} - \frac{y}{4} = 1$.

3. the point $(-3, -4)$ upon the straight line
$$12(x+6) = 5(y-2).$$

4. the point (b, a) upon the straight line $\frac{x}{a} - \frac{y}{b} = 1$.

5. Find the length of the perpendicular from the origin upon the straight line joining the two points whose coordinates are
$$(a \cos \alpha,\ a \sin \alpha) \text{ and } (a \cos \beta,\ a \sin \beta).$$

6. Shew that the product of the perpendiculars drawn from the two points $(\pm \sqrt{a^2 - b^2}, 0)$ upon the straight line
$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \text{ is } b^2.$$

7. If p and p' be the perpendiculars from the origin upon the straight lines whose equations are $x \sec \theta + y \operatorname{cosec} \theta = a$ and
$$x \cos \theta - y \sin \theta = a \cos 2\theta,$$
prove that $$4p^2 + p'^2 = a^2.$$

8. Find the distance between the two parallel straight lines
$$y = mx + c \text{ and } y = mx + d.$$

9. What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a?

10. Shew that the perpendiculars let fall from any point of the straight line $2x + 11y = 5$ upon the two straight lines $24x + 7y = 20$ and $4x - 3y = 2$ are equal to each other.

11. Find the perpendicular distance from the origin of the perpendicular from the point (1, 2) upon the straight line

$$x - \sqrt{3}y + 4 = 0.$$

77. *To find the coordinates of the point of intersection of two given straight lines.*

Let the equations of the two straight lines be

$$a_1x + b_1y + c_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$a_2x + b_2y + c_2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and let the straight lines be AL_1 and AL_2 as in the figure of Art. 66.

Since (1) is the equation of AL_1, the coordinates of *any* point on it must satisfy the equation (1). So the coordinates of any point on AL_2 satisfy equation (2).

Now the only point which is common to these two straight lines is their point of intersection A.

The coordinates of this point must therefore satisfy both (1) and (2).

If therefore A be the point (x_1, y_1), we have

$$a_1x_1 + b_1y_1 + c_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and

$$a_2x_1 + b_2y_1 + c_2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Solving (3) and (4) we have (as in Art. 3)

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1},$$

so that the coordinates of the required common point are

$$\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

78. The coordinates of the point of intersection found in the last article are infinite if

$$a_1b_2 - a_2b_1 = 0.$$

But from Art. 67 we know that the two straight lines are parallel if this condition holds.

Hence parallel lines must be looked upon as lines whose point of intersection is at an infinite distance.

79. *To find the condition that three straight lines may meet in a point.*

Let their equations be

$$a_1x + b_1y + c_1 = 0 \text{ ....(1)},$$
$$a_2x + b_2y + c_2 = 0 \text{ ....(2)},$$

and

$$a_3x + b_3y + c_3 = 0 \text{ ....(3)}.$$

By Art. 77 the coordinates of the point of intersection of (1) and (2) are

$$\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ .... (4)}.$$

If the three straight lines meet in a point, the point of intersection of (1) and (2) must lie on (3). Hence the values (4) must satisfy (3), so that

$$a_3 \times \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} + b_3 \times \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} + c_3 = 0,$$

i.e. $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0,$

i.e. $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0 \ldots (5).$

Aliter. If the three straight lines meet in a point let it be (x_1, y_1), so that the values x_1 and y_1 satisfy the equations (1), (2), and (3), and hence

$$a_1x_1 + b_1y_1 + c_1 = 0,$$
$$a_2x_1 + b_2y_1 + c_2 = 0,$$

and

$$a_3x_1 + b_3y_1 + c_3 = 0.$$

The condition that these three equations should hold between the two quantities x_1 and y_1 is, as in Art. 12,

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0,$$

which is the same as equation (5).

80. Another criterion as to whether the three straight lines of the previous article meet in a point is the following.

If any three quantities p, q, and r can be found so that

$$p(a_1x + b_1y + c_1) + q(a_2x + b_2y + c_2) + r(a_3x + b_3y + c_3) = 0$$

identically, then the three straight lines meet in a point.

For in this case we have

$$a_3x + b_3y + c_3 = -\frac{p}{r}(a_1x + b_1y + c_1) - \frac{q}{r}(a_2x + b_2y + c_2) \quad \ldots(1).$$

Now the coordinates of the point of intersection of the first two of the lines make the right-hand side of (1) vanish. Hence the same coordinates make the left-hand side vanish. The point of intersection of the first two therefore satisfies the equation to the third line and all three therefore meet in a point.

81. Ex. 1. *Shew that the three straight lines* $2x - 3y + 5 = 0$, $3x + 4y - 7 = 0$, *and* $9x - 5y + 8 = 0$ *meet in a point.*

If we multiply these three equations by 3, 1, and -1 we have *identically*

$$3(2x - 3y + 5) + (3x + 4y - 7) - (9x - 5y + 8) = 0.$$

The coordinates of the point of intersection of the first two lines make the first two brackets of this equation vanish and hence make the third vanish. The common point of intersection of the first two therefore satisfies the third equation. The three straight lines therefore meet in a point.

Ex. 2. *Prove that the three perpendiculars drawn from the vertices of a triangle upon the opposite sides all meet in a point.*

Let the triangle be ABC and let its angular points be the points

$$(x_1, y_1),\ (x_2, y_2),\ \text{and}\ (x_3, y_3).$$

The equation to BC is $y - y_2 = \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$.

The equation to the perpendicular from A on this straight line is

$$y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1),$$

i.e. $\quad y(y_3 - y_2) + x(x_3 - x_2) = y_1(y_3 - y_2) + x_1(x_3 - x_2) \quad \ldots\ldots\ldots(1).$

So the perpendiculars from B and C on CA and AB are

$$y(y_1 - y_3) + x(x_1 - x_3) = y_2(y_1 - y_3) + x_2(x_1 - x_3) \quad \ldots\ldots\ldots(2),$$

and $\quad y(y_2 - y_1) + x(x_2 - x_1) = y_3(y_2 - y_1) + x_3(x_2 - x_1) \quad \ldots\ldots\ldots(3).$

On adding these three equations their sum identically vanishes. The straight lines represented by them therefore meet in a point.

This point is called the **orthocentre** of the triangle.

82. *To find the equation to any straight line which passes through the intersection of the two straight lines*

$$a_1x + b_1y + c_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1),$$

and $\quad a_2x + b_2y + c_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2).$

If (x_1, y_1) be the common point of the equations (1) and (2) we may, as in Art. 77, find the values of x_1 and y_1, and then the equation to any straight line through it is

$$y - y_1 = m(x - x_1),$$

where m is any quantity whatever.

Aliter. If A be the common point of the two straight lines, then both equations (1) and (2) are satisfied by the coordinates of the point A.

Hence the equation

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0 \ldots\ldots\ldots(3)$$

is satisfied by the coordinates of the common point A, where λ is any arbitrary constant.

But (3), being of the first degree in x and y, always represents a straight line.

It therefore represents a straight line passing through A.

Also the arbitrary constant λ may be so chosen that (3) may fulfil any other condition. It therefore represents any straight line passing through A.

83. Ex. *Find the equation to the straight line which passes through the intersection of the straight lines*

$$2x - 3y + 4 = 0, \quad 3x + 4y - 5 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and is perpendicular to the straight line

$$6x - 7y + 8 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solving the equations (1), the coordinates x_1, y_1 of their common point are given by

$$\frac{x_1}{(-3)(-5) - 4 \times 4} = \frac{y_1}{4 \times 3 - 2 \times (-5)} = \frac{1}{2 \times 4 - 3 \times (-3)} = \frac{1}{17},$$

so that $\qquad x_1 = -\frac{1}{17}$ and $y_1 = \frac{22}{17}$.

The equation of any straight line through this common point is therefore

$$y - \tfrac{22}{17} = m\left(x + \tfrac{1}{17}\right).$$

This straight line is, by Art. 69, perpendicular to (2) if

$$m \times \tfrac{6}{7} = -1, \text{ i.e. if } m = -\tfrac{7}{6}.$$

The required equation is therefore

$$y - \tfrac{22}{17} = -\tfrac{7}{6}\left(x + \tfrac{1}{17}\right),$$

i.e. $$119x + 102y = 125.$$

Aliter. Any straight line through the intersection of the straight lines (1) is

$$2x-3y+4+\lambda(3x+4y-5)=0,$$

i.e. $$(2+3\lambda)x+y(4\lambda-3)+4-5\lambda=0\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This straight line is perpendicular to (2), if

$$6(2+3\lambda)-7(4\lambda-3)=0, \qquad \text{(Art. 69)}$$

i.e. if $$\lambda=\tfrac{33}{10}.$$

The equation (3) is therefore

$$x\left(2+\tfrac{99}{10}\right)+y\left(\tfrac{132}{10}-3\right)+4-\tfrac{165}{10}=0,$$

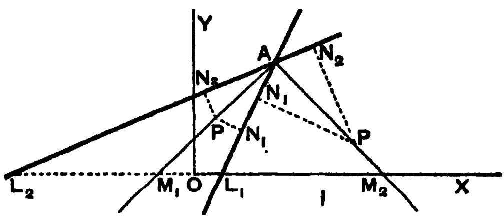
i.e. $$119x+102y-125=0.$$

Bisectors of angles between straight lines.

84. *To find the equations of the bisectors of the angles between the straight lines*

$$a_1x+b_1y+c_1=0\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and $$a_2x+b_2y+c_2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$



Let the two straight lines be AL_1 and AL_2, and let the bisectors of the angles between them be AM_1 and AM_2.

Let P be any point on either of these bisectors and draw PN_1 and PN_2 perpendicular to the given lines.

The triangles PAN_1 and PAN_2 are equal in all respects, so that the perpendiculars PN_1 and PN_2 are equal in magnitude.

Let the equations to the straight lines be written so that c_1 and c_2 are both negative, and to the quantities $\sqrt{a_1^2+b_1^2}$ and $\sqrt{a_2^2+b_2^2}$ let the positive sign be prefixed.

If P be the point (h, k), the numerical values of PN_1 and PN_2 are (by Art. 75)

$$\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}} \text{ and } \frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{.............(1).}$$

If P lie on AM_1, *i.e.* on the bisector of the angle between the two straight lines in which the origin lies, the point P and the origin lie on the same side of each of the two lines. Hence (by Art. 73, Cor.) the two quantities (1) have the same sign as c_1 and c_2 respectively.

In this case, since c_1 and c_2 have the same sign, the quantities (1) have the same sign, and hence

$$\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

But this is the condition that the point (h, k) may lie on the straight line

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}},$$

which is therefore the equation to AM_1.

If, however, P lie on the other bisector AM_2, the two quantities (1) will have opposite signs, so that the equation to AM_2 will be

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

The equations to the original lines being therefore arranged so that the constant terms are both positive (or both negative) the equation to the bisectors is

$$\mathbf{\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}},$$

the upper sign giving the bisector of the angle in which the origin lies.

85. Ex. *Find the equations to the bisectors of the angles between the straight lines*

$$3x - 4y + 7 = 0 \text{ and } 12x - 5y - 8 = 0.$$

Writing the equations so that their constant terms are both positive they are

$$3x - 4y + 7 = 0 \text{ and } -12x + 5y + 8 = 0.$$

The equation to the bisector of the angle in which the origin lies is therefore

$$\frac{3x-4y+7}{\sqrt{3^2+4^2}}=\frac{-12x+5y+8}{\sqrt{12^2+5^2}},$$

i.e. $$13(3x-4y+7)=5(-12x+5y+8),$$

i.e. $$99x-77y+51=0.$$

The equation to the other bisector is

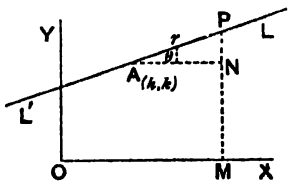
$$\frac{3x-4y+7}{\sqrt{3^2+4^2}}=-\frac{-12x+5y+8}{\sqrt{12^2+5^2}},$$

i.e. $$13(3x-4y+7)+5(-12x+5y+8)=0,$$

i.e. $$21x+27y-131=0.$$

86. It will be found useful in a later chapter to have the equation to a straight line, which passes through a given point and makes a given angle θ with a given line, in a form different from that of Art. 62.

Let A be the given point (h, k) and $L'AL$ a straight line through it inclined at an angle θ to the axis of x.



Take any point P, whose coordinates are (x, y), lying on this line, and let the distance AP be r.

Draw PM perpendicular to the axis of x and AN perpendicular to PM.

Then $$x-h=AN=AP\cos\theta=r\cos\theta,$$

and $$y-k=NP=AP\sin\theta=r\sin\theta.$$

Hence $$\mathbf{\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r} \qquad \text{..........(1).}$$

This being the relation holding between the coordinates of any point P on the line is the equation required.

Cor. From (1) we have

$$x=h+r\cos\theta \text{ and } y=k+r\sin\theta.$$

The coordinates of any point on the given line are therefore $h+r\cos\theta$ and $k+r\sin\theta$.

87. *To find the length of the straight line drawn through a given point in a given direction to meet a given straight line.*

Let the given straight line be

$$Ax + By + C = 0 \text{....................(1).}$$

Let the given point A be (h, k) and the given direction one making an angle θ with the axis of x.

Let the line drawn through A meet the straight line (1) in P and let AP be r.

By the corollary to the last article the coordinates of P are

$$h + r\cos\theta \text{ and } k + r\sin\theta.$$

Since these coordinates satisfy (1) we have

$$A(h + r\cos\theta) + B(k + r\sin\theta) + C = 0.$$

$$\therefore r = -\frac{Ah + Bk + C}{A\cos\theta + B\sin\theta} \text{ .... (2),}$$

giving the length AP which is required.

Cor. From the preceding may be deduced the length of the perpendicular drawn from (h, k) upon (1).

For the "m" of the straight line drawn through A is $\tan\theta$ and the "m" of (1) is $-\frac{A}{B}$.

This straight line is perpendicular to (1) if

$$\tan\theta \times \left(-\frac{A}{B}\right) = -1,$$

i.e. if
$$\tan\theta = \frac{B}{A},$$

so that
$$\frac{\cos\theta}{A} = \frac{\sin\theta}{B} = \frac{1}{\sqrt{A^2 + B^2}},$$

and hence

$$A\cos\theta + B\sin\theta = \frac{A^2 + B^2}{\sqrt{A^2 + B^2}} = \sqrt{A^2 + B^2}.$$

Substituting this value in (2) we have the magnitude of the required perpendicular.

EXAMPLES. VIII.

Find the coordinates of the points of intersection of the straight lines whose equations are

1. $2x - 3y + 5 = 0$ and $7x + 4y = 3$.

2. $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$.

3. $y=m_1x+\frac{a}{m_1}$ and $y=m_2x+\frac{a}{m_2}$.

4. $x\cos\phi_1+y\sin\phi_1=a$ and $x\cos\phi_2+y\sin\phi_2=a$.

5. Two straight lines cut the axis of x at distances a and $-a$ and the axis of y at distances b and b' respectively; find the coordinates of their point of intersection.

6. Find the distance of the point of intersection of the two straight lines

$$2x-3y+5=0 \text{ and } 3x+4y=0$$

from the straight line

$$5x-2y=0.$$

7. Shew that the perpendicular f.om the origin upon the straight line joining the points

$$(a\cos\alpha,\ a\sin\alpha) \text{ and } (a\cos\beta,\ a\sin\beta)$$

bisects the distance between them.

8. Find the equations of the two straight lines drawn through the point $(0, a)$ on which the perpendiculars let fall from the point $(2a, 2a)$ are each of length a.

Prove also that the equation of the straight line joining the feet of these perpendiculars is $y+2x=5a$.

9. Find the point of intersection and the inclination of the two lines

$$Ax+By=A+B \text{ and } A(x-y)+B(x+y)=2B.$$

10. Find the coordinates of the point in which the line

$$2y-3x+7=0$$

meets the line joining the two points $(6, -2)$ and $(-8, 7)$. Find also the angle between them.

11. Find the coordinates of the feet of the perpendiculars let fall from the point $(5, 0)$ upon the sides of the triangle formed by joining the three points $(4, 3)$, $(-4, 3)$, and $(0, -5)$; prove also that the points so determined lie on a straight line.

12. Find the coordinates of the point of intersection of the straight lines

$$2x-3y=1 \text{ and } 5y-x=3,$$

and determine also the angle at which they cut one another.

13. Find the angle between the two lines

$$3x+y+12=0 \text{ and } x+2y-1=0.$$

Find also the coordinates of their point of intersection and the equations of lines drawn perpendicular to them from the point $(3, -2)$.

14. Prove that the points whose coordinates are respectively (5, 1), (1, −1), and (11, 4) lie on a straight line, and find its intercepts on the axes.

Prove that the following sets of three lines meet in a point.

15. $2x-3y=7$, $3x-4y=13$, and $8x-11y=33$.

16. $3x+4y+6=0$, $6x+5y+9=0$, and $3x+3y+5=0$.

17. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, and $y=x$.

18. Prove that the three straight lines whose equations are

$$15x-18y+1=0,\ 12x+10y-3=0,\ \text{and}\ 6x+66y-11=0$$

all meet in a point.

Shew also that the third line bisects the angle between the other two.

19. Find the conditions that the straight lines

$$y=m_1x+a_1,\ y=m_2x+a_2,\ \text{and}\ y=m_3x+a_3$$

may meet in a point.

Find the coordinates of the orthocentre of the triangles whose angular points are

20. (0, 0), (2, −1), and (−1, 3).

21. (1, 0), (2, −4), and (−5, −2).

22. In any triangle ABC, prove that

(1) the bisectors of the angles A, B, and C meet in a point,

(2) the medians, *i.e.* the lines joining each vertex to the middle point of the opposite side, meet in a point,

and (3) the straight lines through the middle points of the sides perpendicular to the sides meet in a point.

Find the equation to the straight line passing through

23. the point (3, 2) and the point of intersection of the lines

$$2x+3y=1\ \text{and}\ 3x-4y=6.$$

24. the point (2, −9) and the intersection of the lines

$$2x+5y-8=0\ \text{and}\ 3x-4y=35.$$

25. the origin and the point of intersection of

$$x-y-4=0\ \text{and}\ 7x+y+20=0,$$

proving that it bisects the angle between them.

26. the origin and the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1\ \text{and}\ \frac{x}{b}+\frac{y}{a}=1.$$

27. the point (a, b) and the intersection of the same two lines.

28. the intersection of the lines

$$x-2y-a=0\ \text{and}\ x+3y-2a=0$$

and parallel to the straight line

$$3x+4y=0.$$

29. the intersection of the lines

$$x+2y+3=0 \text{ and } 3x+4y+7=0$$

and perpendicular to the straight line

$$y-x=8.$$

30. the intersection of the lines

$$3x-4y+1=0 \text{ and } 5x+y-1=0$$

and cutting off equal intercepts from the axes.

31. the intersection of the lines

$$2x-3y=10 \text{ and } x+2y=6$$

and the intersection of the lines

$$16x-10y=33 \text{ and } 12x+14y+29=0.$$

32. If through the angular points of a triangle straight lines be drawn parallel to the sides, and if the intersections of these lines be joined to the opposite angular points of the triangle, show that the joining lines so obtained will meet in a point.

33. Find the equations to the straight lines passing through the point of intersection of the straight lines

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0 \text{ and}$$

(1) passing through the origin,

(2) parallel to the axis of y,

(3) cutting off a given distance a from the axis of y,

and (4) passing through a given point (x', y').

34. Prove that the diagonals of the parallelogram formed by the four straight lines

$$\sqrt{3}x+y=0,\ \sqrt{3}y+x=0,\ \sqrt{3}x+y=1, \text{ and } \sqrt{3}y+x=1$$

are at right angles to one another.

35. Prove the same property for the parallelogram whose sides are

$$\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{b}+\frac{y}{a}=1,\ \frac{x}{a}+\frac{y}{b}=2, \text{ and } \frac{x}{b}+\frac{y}{a}=2.$$

36. One side of a square is inclined to the axis of x at an angle α and one of its extremities is at the origin; prove that the equations to its diagonals are

$$y(\cos\alpha-\sin\alpha)=x(\sin\alpha+\cos\alpha)$$

and

$$y(\sin\alpha+\cos\alpha)+x(\cos\alpha-\sin\alpha)=a$$

where a is the length of the side of the square.

Find the equations to the straight lines bisecting the angles between the following pairs of straight lines, placing first the bisector of the angle in which the origin lies.

37. $x+y\sqrt{3}=6+2\sqrt{3}$ and $x-y\sqrt{3}=6-2\sqrt{3}$.

38. $12x+5y-4=0$ and $3x+4y+7=0$.

39. $4x+3y-7=0$ and $24x+7y-31=0$.

40. $2x+y=4$ and $y+3x=5$.

41. Find the bisectors of the angles between the straight lines

$$y-b=\frac{2m}{1-m^2}(x-a) \text{ and } y-b=\frac{2m'}{1-m'^2}(x-a).$$

Find the equations to the bisectors of the internal angles of the triangles the equations of whose sides are respectively

42. $3x+4y=6$, $12x-5y=3$, and $4x-3y+12=0$.

43. $3x+5y=15$, $x+y=4$, and $2x+y=6$.

44. Find the equations to the straight lines passing through the foot of the perpendicular from the point (h, k) upon the straight line $Ax+By+C=0$ and bisecting the angles between the perpendicular and the given straight line.

45. Find the direction in which a straight line must be drawn through the point $(1, 2)$, so that its point of intersection with the line $x+y=4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.

CHAPTER V.

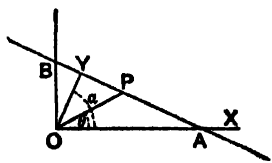
THE STRAIGHT LINE (*continued*).

POLAR EQUATIONS. OBLIQUE COORDINATES.

MISCELLANEOUS PROBLEMS. LOCI.

88. *To find the general equation to a straight line in polar coordinates.*

Let p be the length of the perpendicular OY from the origin upon the straight line, and let this perpendicular make an angle α with the initial line.



Let P be any point on the line and let its coordinates be r and θ.

The equation required will then be the relation between r, θ, p, and α.

From the triangle OYP we have

$$p = r\cos YOP = r\cos(\alpha - \theta) = r\cos(\theta - \alpha).$$

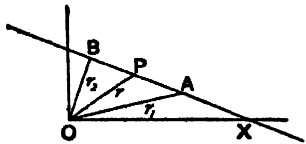
The required equation is therefore

$$r\cos(\theta - \alpha) = p.$$

[On transforming to Cartesian coordinates this equation becomes the equation of Art. 53.]

89. *To find the polar equation of the straight line joining the points whose coordinates are* (r_1, θ_1) *and* (r_2, θ_2).

Let A and B be the two given points and P any point on the line joining them whose coordinates are r and θ.



Then, since

$$\triangle AOB = \triangle AOP + \triangle POB,$$

we have

$$\tfrac{1}{2} r_1 r_2 \sin AOB = \tfrac{1}{2} r_1 r \sin AOP + \tfrac{1}{2} r r_2 \sin POB,$$

i.e. $$r_1 r_2 \sin(\theta_2 - \theta_1) = r_1 r \sin(\theta - \theta_1) + r r_2 \sin(\theta_2 - \theta),$$

i.e. $$\frac{\sin(\theta_2 - \theta_1)}{r} = \frac{\sin(\theta - \theta_1)}{r_2} + \frac{\sin(\theta_2 - \theta)}{r_1}.$$

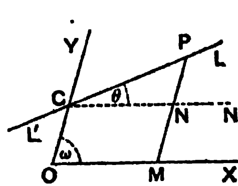
OBLIQUE COORDINATES.

90. In the previous chapter we took the axes to be rectangular. In the great majority of cases rectangular axes are employed, but in some cases oblique axes may be used with advantage.

In the following articles we shall consider the propositions in which the results for oblique axes are different from those for rectangular axes. The propositions of Arts. 50 and 62 are true for oblique, as well as rectangular, coordinates.

91. *To find the equation to a straight line referred to axes inclined at an angle ω.*

Let LPL' be a straight line which cuts the axis of Y at a distance c from the origin and is inclined at an angle θ to the axis of x.



Let P be any point on the straight line. Draw PNM parallel to the axis of y to meet OX in M, and let it meet the straight line through C parallel to the axis of x in the point N.

Let P be the point (x, y), so that

$$CN = OM = x, \text{ and } NP = MP - OC = y - c.$$

Since $\angle CPN = \angle PNN' - \angle PCN' = \omega - \theta$, we have

$$\frac{y-c}{x} = \frac{NP}{CN} = \frac{\sin NCP}{\sin CPN} = \frac{\sin\theta}{\sin(\omega-\theta)}.$$

Hence $$y = x\frac{\sin\theta}{\sin(\omega-\theta)} + c \qquad \text{.................(1).}$$

This equation is of the form

$$y = mx + c,$$

where

$$m = \frac{\sin\theta}{\sin(\omega-\theta)} = \frac{\sin\theta}{\sin\omega\cos\theta - \cos\omega\sin\theta} = \frac{\tan\theta}{\sin\omega - \cos\omega\tan\theta},$$

and therefore $$\tan\theta = \frac{m\sin\omega}{1+m\cos\omega}.$$

In oblique coordinates the equation

$$y = mx + c$$

therefore represents a straight line which is inclined at an angle

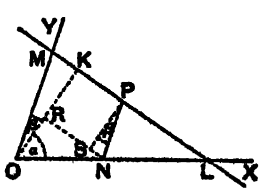
$$\tan^{-1}\frac{m\sin\omega}{1+m\cos\omega}$$

to the axis of x.

Cor. From (1), by putting in succession θ equal to $90°$ and $90° + \omega$, we see that the equations to the straight lines, passing through the origin and perpendicular to the axes of x and y, are respectively $y = -\frac{x}{\cos\omega}$ and $y = -x\cos\omega$.

92. *The axes being oblique, to find the equation to the straight line, such that the perpendicular on it from the origin is of length p and makes angles α and β with the axes of x and y.*

Let LM be the given straight line and OK the perpendicular on it from the origin.



Let P be *any* point on the straight line; draw the ordinate PN and draw NR perpendicular to OK and PS perpendicular to NR.

Let P be the point (x, y), so that $ON = x$ and $NP = y$.

The lines NP and OY are parallel.

Also OK and SP are parallel, each being perpendicular to NR.

Thus $\angle SPN = \angle KOM = \beta$.

We therefore have

$$p = OK = OR + SP = ON\cos\alpha + NP\cos\beta = x\cos\alpha + y\cos\beta.$$

Hence $x\cos\alpha + y\cos\beta - p = 0$,

being the relation which holds between the coordinates of any point on the straight line, is the required equation.

93. *To find the angle between the straight lines*

$$y = mx + c \text{ and } y = m'x + c',$$

the axes being oblique.

If these straight lines be respectively inclined at angles θ and θ' to the axis of x, we have, by the last article,

$$\tan\theta = \frac{m\sin\omega}{1 + m\cos\omega} \text{ and } \tan\theta' = \frac{m'\sin\omega}{1 + m'\cos\omega}.$$

The angle required is $\theta \sim \theta'$.

Now $$\tan(\theta - \theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta . \tan\theta'}$$

$$= \frac{\dfrac{m\sin\omega}{1 + m\cos\omega} - \dfrac{m'\sin\omega}{1 + m'\cos\omega}}{1 + \dfrac{m\sin\omega}{1 + m\cos\omega}\dfrac{m'\sin\omega}{1 + m'\cos\omega}}$$

$$= \frac{m\sin\omega(1 + m'\cos\omega) - m'\sin\omega(1 + m\cos\omega)}{(1 + m\cos\omega)(1 + m'\cos\omega) + mm'\sin^2\omega}$$

$$= \frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

The required angle is therefore

$$\tan^{-1}\frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

Cor. 1. The two given lines are parallel if $\mathbf{m = m'}$.

Cor. 2. The two given lines are perpendicular if

$$\mathbf{1 + (m + m')\cos\omega + mm' = 0}.$$

94. If the straight lines have their equations in the form

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0,$$

then
$$m = -\frac{A}{B} \text{ and } m' = -\frac{A'}{B'}.$$

Substituting these values in the result of the last article the angle between the two lines is easily found to be

$$\tan^{-1} \frac{A'B - AB'}{AA' + BB' - (AB' + A'B)\cos\omega} \sin\omega.$$

The given lines are therefore parallel if

$$A'B - AB' = 0.$$

They are perpendicular if

$$AA' + BB' = (AB' + A'B)\cos\omega.$$

95. Ex. *The axes being inclined at an angle of 30°, obtain the equations to the straight lines which pass through the origin and are inclined at 45° to the straight line $x + y = 1$.*

Let either of the required straight lines be $y = mx$.

The given straight line is $y = -x + 1$, so that $m' = -1$.

We therefore have

$$\frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'} = \tan(\pm 45°),$$

where $m' = -1$ and $\omega = 30°$.

This equation gives $\dfrac{m + 1}{2 + (m - 1)\sqrt{3} - 2m} = \pm 1.$

Taking the upper sign we obtain $m = -\dfrac{1}{\sqrt{3}}$.

Taking the lower sign we have $m = -\sqrt{3}$.

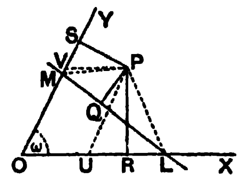
The required equations are therefore

$$y = -\sqrt{3}x \text{ and } y = -\frac{1}{\sqrt{3}}x,$$

i.e.
$$y + \sqrt{3}x = 0 \text{ and } \sqrt{3}y + x = 0.$$

96. *To find the length of the perpendicular from the point (x', y') upon the straight line $Ax + By + C = 0$, the axes being inclined at an angle ω, and the equation being written so that C is a negative quantity.*

Let the given straight line meet the axes in L and M, so that $OL = -\frac{C}{A}$ and $OM = -\frac{C}{B}$.



Let P be the given point (x', y'). Draw the perpendiculars PQ, PR, and PS on the given line and the two axes.

Taking O and P on opposite sides of the given line, we then have

$$\triangle LPM + \triangle MOL = \triangle OLP + \triangle OPM,$$

i.e. $\quad PQ \cdot LM + OL \cdot OM \sin\omega = OL \cdot PR + OM \cdot PS \ldots (1).$

Draw PU and PV parallel to the axes of y and x, so that $PU = y'$ and $PV = x'$.

Hence $\qquad PR = PU \sin PUR = y' \sin\omega,$

and $\qquad PS = PV \sin PVS = x' \sin\omega.$

Also

$$LM = \sqrt{OL^2 + OM^2 - 2OL \cdot OM \cos\omega}$$

$$= \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2} - 2\frac{C^2}{AB}\cos\omega} = -C\sqrt{\frac{1}{A^2} + \frac{1}{B^2} - \frac{2\cos\omega}{AB}},$$

since C is a negative quantity.

On substituting these values in (1), we have

$$PQ \times (-C) \times \sqrt{\frac{1}{A^2} + \frac{1}{B^2} - \frac{2\cos\omega}{AB}} + \frac{C^2}{AB}\sin\omega$$

$$= -\frac{C}{A} \cdot y' \sin\omega - \frac{C}{B} \cdot x' \sin\omega,$$

so that $\qquad PQ = \dfrac{\mathbf{Ax' + By' + C}}{\sqrt{\mathbf{A^2 + B^2 - 2AB\cos\omega}}} \cdot \mathbf{\sin\omega}.$

Cor. If $\omega = 90^\circ$, *i.e.* if the axes be rectangular, we have the result of Art. 75.

EXAMPLES. IX.

1. The axes being inclined at an angle of 60°, find the inclination to the axis of x of the straight lines whose equations are

(1) $y=2x+5$,

and (2) $2y=(\sqrt{3}-1)x+7$.

2. The axes being inclined at an angle of 120°, find the tangent of the angle between the two straight lines

$$8x+7y=1 \text{ and } 28x-73y=101.$$

3. With oblique coordinates find the tangent of the angle between the straight lines

$$y=mx+c \text{ and } my+x=d.$$

4. If $y=x\tan\frac{11\pi}{24}$ and $y=x\tan\frac{19\pi}{24}$ represent two straight lines at right angles, prove that the angle between the axes is $\frac{\pi}{4}$.

5. Prove that the straight lines $y+x=c$ and $y=x+d$ are at right angles, whatever be the angle between the axes.

6. Prove that the equation to the straight line which passes through the point (h, k) and is perpendicular to the axis of x is

$$x+y\cos\omega=h+k\cos\omega.$$

7. Find the equations to the sides and diagonals of a regular hexagon, two of its sides, which meet in a corner, being the axes of coordinates.

8. From each corner of a parallelogram a perpendicular is drawn upon the diagonal which does not pass through that corner and these are produced to form another parallelogram; shew that its diagonals are perpendicular to the sides of the first parallelogram and that they both have the same centre.

9. If the straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ make equal angles with the axis of x and be not parallel to one another, prove that $m_1+m_2+2m_1m_2\cos\omega=0$.

10. The axes being inclined at an angle of 30°, find the equation to the straight line which passes through the point $(-2, 3)$ and is perpendicular to the straight line $y+3x=6$.

11. Find the length of the perpendicular drawn from the point $(4, -3)$ upon the straight line $6x+3y-10=0$, the angle between the axes being 60°.

12. Find the equation to, and the length of, the perpendicular drawn from the point $(1, 1)$ upon the straight line $3x+4y+5=0$, the angle between the axes being 120°.

13. The coordinates of a point P referred to axes meeting at an angle ω are (h, k); prove that the length of the straight line joining the feet of the perpendiculars from P upon the axes is

$$\sin\omega\sqrt{h^2+k^2+2hk\cos\omega}.$$

14. From a given point (h, k) perpendiculars are drawn to the axes, whose inclination is ω, and their feet are joined. Prove that the length of the perpendicular drawn from (h, k) upon this line is

$$\frac{hk\sin^2\omega}{\sqrt{h^2+k^2+2hk\cos\omega}},$$

and that its equation is $hx-ky=h^2-k^2$.

Straight lines passing through fixed points.

97. *If the equation to a straight line be of the form*

$$ax+by+c+\lambda(a'x+b'y+c')=0 \quad \text{.........(1)},$$

where λ is any arbitrary constant, it always passes through one fixed point whatever be the value of λ.

For the equation (1) is satisfied by the coordinates of the point which satisfies *both* of the equations

$$ax+by+c=0,$$

and

$$a'x+b'y+c'=0.$$

This point is, by Art. 77,

$$\left(\frac{bc'-b'c}{ab'-a'b},\ \frac{ca'-c'a}{ab'-a'b}\right),$$

and these coordinates are independent of λ.

Ex. *Given the vertical angle of a triangle in magnitude and position, and also the sum of the reciprocals of the sides which contain it; shew that the base always passes through a fixed point.*

Take the fixed angular point as origin and the directions of the sides containing it as axes; let the lengths of these sides in any such triangle be a and b, which are not therefore given.

We have $\qquad \frac{1}{a}+\frac{1}{b}=\text{const.}=\frac{1}{k}$ (say)........................(1).

The equation to the base is

$$\frac{x}{a}+\frac{y}{b}=1,$$

i.e., by (1), $\qquad \frac{x}{a}+y\left(\frac{1}{k}-\frac{1}{a}\right)=1,$

i.e. $\qquad \frac{1}{a}(x-y)+\frac{y}{k}-1=0.$

Whatever be the value of a this straight line always passes through the point given by

$$x-y=0 \text{ and } \frac{y}{k}-1=0,$$

i.e. through the *fixed* point (k, k).

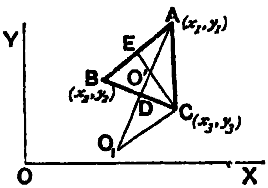
98. *Prove that the coordinates of the centre of the circle inscribed in the triangle, whose vertices are the points* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3), *are*

$$\frac{ax_1+bx_2+cx_3}{a+b+c} \text{ and } \frac{ay_1+by_2+cy_3}{a+b+c},$$

where a, b, and c are the lengths of the sides of the triangle.

Find also the coordinates of the centres of the escribed circles.

Let ABC be the triangle and let AD and CE be the bisectors of the angles A and C and let them meet in O'.



Then O' is the required point.

Since AD bisects the angle BAC we have, by geometry,

$$\frac{BD}{BA}=\frac{DC}{AC}=\frac{BD+DC}{BA+AC}=\frac{a}{b+c},$$

so that

$$DC=\frac{ba}{b+c}.$$

Also, since CO' bisects the angle ACD, we have

$$\frac{AO'}{O'D}=\frac{AC}{CD}=\frac{b}{\frac{ba}{b+c}}=\frac{b+c}{a}.$$

The point D therefore divides BC in the ratio

$$BA : AC, \text{ i.e. } c : b.$$

Also O' divides AD in the ratio $b+c : a$.

Hence, by Art. 22, the coordinates of D are

$$\frac{cx_3+bx_2}{c+b} \text{ and } \frac{cy_3+by_2}{c+b}.$$

Also, by the same article, the coordinates of O' are

$$\frac{(b+c)\times\frac{cx_3+bx_2}{c+b}+ax_1}{(b+c)+a} \text{ and } \frac{(b+c)\times\frac{cy_3+by_2}{c+b}+ay_1}{(b+c)+a},$$

i.e. $$\frac{ax_1+bx_2+cx_3}{a+b+c} \text{ and } \frac{ay_1+by_2+cy_3}{a+b+c}.$$

Again, if O_1 be the centre of the escribed circle opposite to the angle A, the line CO_1 bisects the exterior angle of ACB.

Hence, by geometry, we have

$$\frac{AO_1}{O_1D}=\frac{AC}{CD}=\frac{b+c}{a}.$$

Therefore O_1 is the point which divides AD *externally* in the ratio $b+c:a$.

Its coordinates (Art. 22) are therefore

$$\frac{(b+c)\frac{cx_3+bx_2}{c+b}-ax_1}{(b+c)-a} \text{ and } \frac{(b+c)\frac{cy_3+by_2}{c+b}-ay_1}{(b+c)-a},$$

i.e. $$\frac{-ax_1+bx_2+cx_3}{-a+b+c} \text{ and } \frac{-ay_1+by_2+cy_3}{-a+b+c}.$$

Similarly, it may be shewn that the coordinates of the centres of the escribed circles opposite to B and C are respectively

$$\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\ \frac{ay_1-by_2+cy_3}{a-b+c}\right),$$

and $$\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\ \frac{ay_1+by_2-cy_3}{a+b-c}\right).$$

99. As a numerical example consider the case of the triangle formed by the straight lines

$$3x+4y-7=0,\quad 12x+5y-17=0 \text{ and } 5x+12y-34=0.$$

These three straight lines being BC, CA, and AB respectively we easily obtain, by solving, that the points A, B, and C are

$$\left(\frac{2}{7},\ \frac{19}{7}\right),\ \left(\frac{-52}{16},\ \frac{67}{16}\right) \text{ and } (1,\ 1).$$

Hence

$$a=\sqrt{\left(\frac{-52}{16}-1\right)^2+\left(\frac{67}{16}-1\right)^2}=\sqrt{\frac{68^2}{16^2}+\frac{51^2}{16^2}}$$

$$=\frac{17}{16}\sqrt{4^2+3^2}=\frac{85}{16},$$

$$b=\sqrt{\left(1-\frac{2}{7}\right)^2+\left(1-\frac{19}{7}\right)^2}=\sqrt{\frac{5^2}{7^2}+\frac{12^2}{7^2}}=\frac{13}{7},$$

and

$$c=\sqrt{\left(\frac{2}{7}+\frac{52}{16}\right)^2+\left(\frac{19}{7}-\frac{67}{16}\right)^2}=\sqrt{\frac{396^2+165^2}{112^2}}$$

$$=\frac{33}{112}\sqrt{169}=\frac{429}{112}.$$

Hence

$$ax_1=\frac{85}{16}\times\frac{2}{7}=\frac{170}{112};\quad ay_1=\frac{85}{16}\times\frac{19}{7}=\frac{1615}{112};$$

$$bx_2=\frac{13}{7}\times\frac{-52}{16}=-\frac{676}{112};\quad by_2=\frac{13}{7}\times\frac{67}{16}=\frac{871}{112};$$

$$cx_3=\frac{429}{112};\quad\text{and}\quad cy_3=\frac{429}{112}.$$

The coordinates of the centre of the incircle are therefore

$$\frac{\frac{170}{112}-\frac{676}{112}+\frac{429}{112}}{\frac{85}{16}+\frac{13}{7}+\frac{429}{112}}\quad\text{and}\quad\frac{\frac{1615}{112}+\frac{871}{112}+\frac{429}{112}}{\frac{85}{16}+\frac{13}{7}+\frac{429}{112}},$$

i.e.

$$\frac{-1}{16}\quad\text{and}\quad\frac{265}{112}.$$

The length of the radius of the incircle is the perpendicular from $\left(-\frac{1}{16},\frac{265}{112}\right)$ upon the straight line

$$3x+4y-7=0,$$

and therefore
$$=\frac{\left(3\times-\frac{1}{16}\right)+\left(4\times\frac{265}{112}\right)-7}{\sqrt{3^2+4^2}}$$

$$=\frac{-21+1060-784}{5\times 112}=\frac{255}{5\times 112}=\frac{51}{112}.$$

The coordinates of the centre of the escribed circle which touches the side BC externally are

$$\frac{-\frac{170}{112}-\frac{676}{112}+\frac{429}{112}}{-\frac{85}{16}+\frac{13}{7}+\frac{429}{112}} \text{ and } \frac{-\frac{1615}{112}+\frac{871}{112}+\frac{429}{112}}{-\frac{85}{16}+\frac{13}{7}+\frac{429}{112}},$$

i.e.
$$\frac{-417}{42} \text{ and } \frac{-315}{42}.$$

Similarly the coordinates of the centres of the other escribed circles can be written down.

100. Ex. *Find the radius, and the coordinates of the centre, of the circle circumscribing the triangle formed by the points*

$$(0, 1), \quad (2, 3), \quad \textit{and} \quad (3, 5).$$

Let (x_1, y_1) be the required centre and R the radius.

Since the distance of the centre from each of the three points is the same, we have

$$x_1^2+(y_1-1)^2=(x_1-2)^2+(y_1-3)^2=(x_1-3)^2+(y_1-5)^2=R^2 \ldots (1).$$

From the first two we have, on reduction,

$$x_1+y_1=3.$$

From the first and third equations we obtain

$$6x_1+8y_1=33.$$

Solving, we have $x_1=-\frac{9}{2}$ and $y_1=\frac{15}{2}$.

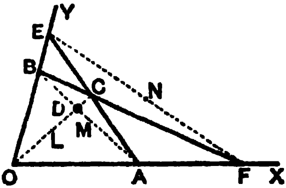
Substituting these values in (1) we get

$$R=\tfrac{5}{2}\sqrt{10}.$$

101. Ex. *Prove that the middle points of the diagonals of a complete quadrilateral lie on the same straight line.*

[**Complete quadrilateral. Def.** Let $OACB$ be any quadrilateral. Let AC and OB be produced to meet in E, and BC and OA to meet in F. Join AB, OC, and EF. The resulting figure is called a complete quadrilateral; the lines AB, OC, and EF are called its diagonals, and the points E, F, and D (the intersection of AB and OC) are called its vertices.]

Take the lines *OAF* and *OBE* as the axes of x and y.



Let $OA=2a$ and $OB=2b$, so that A is the point $(2a, 0)$ and B is the point $(0, 2b)$; also let C be the point $(2h, 2k)$.

Then L, the middle point of OC, is the point (h, k), and M, the middle point of AB, is (a, b).

The equation to LM is therefore

$$y-b=\frac{k-b}{h-a}(x-a),$$

i.e.

$$(h-a)\,y-(k-b)\,x=bh-ak \qquad \text{.........(1)}.$$

Again, the equation to BC is $y-2b=\frac{k-b}{h}x$.

Putting $y=0$, we have $x=\frac{-2bh}{k-b}$, so that F is the point

$$\left(\frac{-2bh}{k-b},\, 0\right).$$

Similarly, E is the point $\left(0,\ -\frac{2ak}{h-a}\right)$.

Hence N, the middle point of EF, is $\left(\frac{-bh}{k-b},\ \frac{-ak}{h-a}\right)$.

These coordinates clearly satisfy (1), *i.e.* N lies on the straight line LM.

EXAMPLES. X.

1. A straight line is such that the algebraic sum of the perpendiculars let fall upon it from any number of fixed points is zero; shew that it always passes through a fixed point.

2. Two fixed straight lines OX and OY are cut by a variable line in the points A and B respectively and P and Q are the feet of the perpendiculars drawn from A and B upon the lines OBY and OAX. Shew that, if AB pass through a fixed point, then PQ will also pass through a fixed point.

3. If the equal sides AB and AC of an isosceles triangle be produced to E and F so that $BE \cdot CF = AB^2$, shew that the line EF will always pass through a fixed point.

4. If a straight line move so that the sum of the perpendiculars let fall on it from the two fixed points (3, 4) and (7, 2) is equal to three times the perpendicular on it from a third fixed point (1, 3), prove that there is another fixed point through which this line always passes and find its coordinates.

Find the centre and radius of the circle which is inscribed in the triangle formed by the straight lines whose equations are

5. $3x+4y+2=0$, $3x-4y+12=0$, and $4x-3y=0$.

6. $2x+4y+3=0$, $4x+3y+3=0$, and $x+1=0$.

7. $y=0$, $12x-5y=0$, and $3x+4y-7=0$.

8. Prove that the coordinates of the centre of the circle inscribed in the triangle whose angular points are (1, 2), (2, 3), and (3, 1) are $\frac{8+\sqrt{10}}{6}$ and $\frac{16-\sqrt{10}}{6}$.

Find also the coordinates of the centres of the escribed circles.

9. Find the coordinates of the centres, and the radii, of the four circles which touch the sides of the triangle the coordinates of whose angular points are the points (6, 0), (0, 6), and (7, 7).

10. Find the position of the centre of the circle circumscribing the triangle whose vertices are the points (2, 3), (3, 4), and (6, 8).

Find the area of the triangle formed by the straight lines whose equations are

11. $y=x$, $y=2x$, and $y=3x+4$.

12. $y+x=0$, $y=x+6$, and $y=7x+5$.

13. $2y+x-5=0$, $y+2x-7=0$, and $x-y+1=0$.

14. $3x-4y+4a=0$, $2x-3y+4a=0$, and $5x-y+a=0$, proving also that the feet of the perpendiculars from the origin upon them are collinear.

15. $y=ax-bc$, $y=bx-ca$, and $y=cx-ab$.

16. $y=m_1x+\frac{a}{m_1}$, $y=m_2x+\frac{a}{m_2}$, and $y=m_3x+\frac{a}{m_3}$.

17. $y=m_1x+c_1$, $y=m_2x+c_2$, and the axis of y.

18. $y=m_1x+c_1$, $y=m_2x+c_2$, and $y=m_3x+c_3$.

19. Prove that the area of the triangle formed by the three straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, and $a_3x+b_3y+c_3=0$ is

$$\frac{1}{2}\left\{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}\right\}^2 \div (a_1b_2-a_2b_1)(a_2b_3-a_3b_2)(a_3b_1-a_1b_3).$$

20. Prove that the area of the triangle formed by the three straight lines

$$x\cos\alpha + y\sin\alpha - p_1 = 0, \quad x\cos\beta + y\sin\beta - p_2 = 0,$$

and

$$x\cos\gamma + y\sin\gamma - p_3 = 0,$$

is

$$\frac{1}{2}\frac{\{p_1\sin(\gamma-\beta) + p_2\sin(\alpha-\gamma) + p_3\sin(\beta-\alpha)\}^2}{\sin(\gamma-\beta)\sin(\alpha-\gamma)\sin(\beta-\alpha)}.$$

21. Prove that the area of the parallelogram contained by the lines

$$4y - 3x - a = 0,\ 3y - 4x + a = 0,\ 4y - 3x - 3a = 0,$$

and

$$3y - 4x + 2a = 0 \text{ is } \tfrac{2}{7}a^2.$$

22. Prove that the area of the parallelogram whose sides are the straight lines

$$a_1x + b_1y + c_1 = 0,\ a_1x + b_1y + d_1 = 0,\ a_2x + b_2y + c_2 = 0,$$

and

$$a_2x + b_2y + d_2 = 0$$

is

$$\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}.$$

23. The vertices of a quadrilateral, taken in order, are the points (0, 0), (4, 0), (6, 7), and (0, 3); find the coordinates of the point of intersection of the two lines joining the middle points of opposite sides.

24. The lines $x + y + 1 = 0$, $x - y + 2 = 0$, $4x + 2y + 3 = 0$, and

$$x + 2y - 4 = 0$$

are the equations to the sides of a quadrilateral taken in order; find the equations to its three diagonals and the equation to the line on which their middle points lie.

25. Shew that the orthocentre of the triangle formed by the three straight lines

$$y = m_1x + \frac{a}{m_1}, \quad y = m_2x + \frac{a}{m_2}, \text{ and } y = m_3x + \frac{a}{m_3}$$

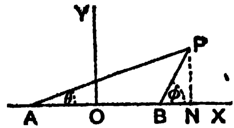
is the point

$$\left\{-a,\ a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)\right\}.$$

26. A and B are two fixed points whose coordinates are (3, 2) and (5, 1) respectively; ABP is an equilateral triangle on the side of AB remote from the origin. Find the coordinates of P and the orthocentre of the triangle ABP.

102. Ex. *The base of a triangle is fixed; find the locus of the vertex when one base angle is double of the other.*

Let AB be the fixed base of the triangle; take its middle point O as origin, the direction of OB as the axis of x and a perpendicular line as the axis of y.



Let $AO = OB = a$.

If P be one position of the vertex, the condition of the problem then gives

$$\angle PBA = 2\angle PAB,$$

i.e. $$\pi - \phi = 2\theta,$$

i.e. $$-\tan\phi = \tan 2\theta \qquad\text{.....................(1).}$$

Let P be the point (h, k). We then have

$$\frac{k}{h+a} = \tan\theta \text{ and } \frac{k}{h-a} = \tan\phi.$$

Substituting these values in (1), we have

$$-\frac{k}{h-a} = \frac{2\dfrac{k}{h+a}}{1-\left(\dfrac{k}{h+a}\right)^2} = \frac{2(h+a)k}{(h+a)^2-k^2},$$

i.e. $$-(h+a)^2 + k^2 = 2(h^2-a^2),$$

i.e. $$k^2 - 3h^2 - 2ah + a^2 = 0.$$

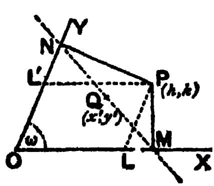
But this is the condition that the point (h, k) should lie on the curve

$$y^2 - 3x^2 - 2ax + a^2 = 0.$$

This is therefore the equation to the required locus.

103. Ex. *From a point P perpendiculars PM and PN are drawn upon two fixed lines which are inclined at an angle ω and meet in a fixed point O; if P move on a fixed straight line, find the locus of the middle point of MN.*

Let the two fixed lines be taken as the axes. Let the coordinates of P, any position of the moving point, be (h, k).



Let the equation of the straight line on which P lies be

$$Ax + By + C = 0,$$

so that we have

$$Ah + Bk + C = 0 \text{ ...... (1).}$$

Draw PL and PL' parallel to the axes.

We then have

$$OM = OL + LM = OL + LP \cos \omega = h + k \cos \omega,$$

and $$ON = OL' + L'N = LP + L'P \cos \omega = k + h \cos \omega.$$

M is therefore the point $(h + k \cos \omega,\ 0)$ and N is the point $(0,\ k + h \cos \omega)$.

Hence, if (x', y') be the coordinates of the middle point of MN, we have

$$2x' = h + k \cos \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

and $$2y' = k + h \cos \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Equations (1), (2), and (3) express analytically all the relations which hold between x', y', h, and k.

Also h and k are the quantities which by their variation cause Q to take up different positions. If therefore between (1), (2), and (3) we eliminate h and k we shall obtain a relation between x' and y' which is true for *all* values of h and k, *i.e* a relation which is true whatever be the position that P takes on the given straight line.

From (2) and (3), by solving, we have

$$h = \frac{2(x' - y' \cos \omega)}{\sin^2 \omega} \text{ and } k = \frac{2(y' - x' \cos \omega)}{\sin^2 \omega}.$$

Substituting these values in (1), we obtain

$$2A(x' - y' \cos \omega) + 2B(y' - x' \cos \omega) + C \sin^2 \omega = 0.$$

But this is the condition that the point (x', y') shall always lie on the straight line

$$2A(x - y \cos \omega) + 2B(y - x \cos \omega) + C \sin^2 \omega = 0,$$

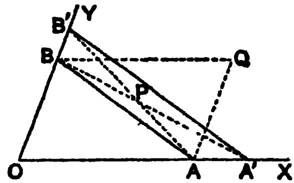
i.e. on the straight line

$$x(A - B \cos \omega) + y(B - A \cos \omega) + \tfrac{1}{2} C \sin^2 \omega = 0,$$

which is therefore the equation to the locus of Q.

104. Ex. *A straight line is drawn parallel to the base of a given triangle and its extremities are joined transversely to those of the base; find the locus of the point of intersection of the joining lines.*

Let the triangle be OAB and take O as the origin and the directions of OA and OB as the axes of x and y.



Let $OA = a$ and $OB = b$, so that a and b are given quantities.

Let $A'B'$ be the straight line which is parallel to the base AB; so that

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \lambda \text{ (say)},$$

and hence $\qquad OA' = \lambda a$ and $OB' = \lambda b$.

For different values of λ we therefore have different positions of $A'B'$.

The equation to AB' is

$$\frac{x}{a} + \frac{y}{\lambda b} = 1 \qquad \text{.......................(1)},$$

and that to $A'B$ is

$$\frac{x}{\lambda a} + \frac{y}{b} = 1 \qquad \text{.......................(2)}.$$

Since P is the intersection of AB' and $A'B$ its coordinates satisfy both (1) and (2). Whatever equation we derive from them must therefore denote a locus going through P. Also if we derive from (1) and (2) an equation which does not contain λ, it must represent a locus which passes through P whatever be the value of λ; in other words it must go through all the different positions of the point P.

Subtracting (2) from (1), we have

$$\frac{x}{a}\left(1 - \frac{1}{\lambda}\right) + \frac{y}{b}\left(\frac{1}{\lambda} - 1\right) = 0,$$

i.e. $$\frac{x}{a} = \frac{y}{b}.$$

This then is the equation to the locus of P. Hence P always lies on the straight line

$$y = \frac{b}{a}x,$$

which is the straight line OQ where $OAQB$ is a parallelogram.

Aliter. By solving the equations (1) and (2) we easily see that they meet at the point

$$\left(\frac{\lambda}{\lambda+1}a,\ \frac{\lambda}{\lambda+1}b\right).$$

Hence, if P be the point (h, k), we have

$$h=\frac{\lambda}{\lambda+1}a \text{ and } k=\frac{\lambda}{\lambda+1}b.$$

Hence for all values of λ, *i.e.* for all positions of the straight line $A'B'$, we have

$$\frac{h}{a}=\frac{k}{b}.$$

But this is the condition that the point (h, k), *i.e.* P, should lie on the straight line

$$\frac{x}{a}=\frac{y}{b}.$$

The straight line is therefore the required locus.

105. Ex. *A variable straight line is drawn through a given point O to cut two fixed straight lines in R and S; on it is taken a point P such that*

$$\frac{2}{OP}=\frac{1}{OR}+\frac{1}{OS};$$

shew that the locus of P is a third fixed straight line.

Take any two fixed straight lines, at right angles and passing through O, as the axes and let the equation to the two given fixed straight lines be

$$Ax+By+C=0,$$

and

$$A'x+B'y+C'=0.$$

Transforming to polar coordinates these equations are

$$\frac{1}{r}=-\frac{A\cos\theta+B\sin\theta}{C} \text{ and } \frac{1}{r}=-\frac{A'\cos\theta+B'\sin\theta}{C'}.$$

If the angle XOR be θ the values of $\frac{1}{OR}$ and $\frac{1}{OS}$ are therefore

$$-\frac{A\cos\theta + B\sin\theta}{C} \text{ and } -\frac{A'\cos\theta + B'\sin\theta}{C'}.$$

We therefore have

$$\frac{2}{OP} = -\frac{A\cos\theta + B\sin\theta}{C} - \frac{A'\cos\theta + B'\sin\theta}{C'}$$

$$= -\left(\frac{A}{C} + \frac{A'}{C'}\right)\cos\theta - \left(\frac{B}{C} + \frac{B'}{C'}\right)\sin\theta.$$

The equation to the locus of P is therefore, on again transforming to Cartesian coordinates,

$$2 = -x\left(\frac{A}{C} + \frac{A'}{C'}\right) - y\left(\frac{B}{C} + \frac{B'}{C'}\right),$$

and this is a fixed straight line.

EXAMPLES. XI.

The base BC $(=2a)$ of a triangle ABC is fixed; the axes being BC and a perpendicular to it through its middle point, find the locus of the vertex A, when

1. the difference of the base angles is given $(=\alpha)$.

2. the product of the tangents of the base angles is given $(=\lambda)$.

3. the tangent of one base angle is m times the tangent of the other.

4. m times the square of one side added to n times the square of the other side is equal to a constant quantity c^2.

From a point P perpendiculars PM and PN are drawn upon two fixed lines which are inclined at an angle ω, and which are taken as the axes of coordinates and meet in O; find the locus of P

5. if $OM + ON$ be equal to $2c$.

6. if $OM - ON$ be equal to $2d$.

7. if $PM + PN$ be equal to $2c$.

8. if $PM - PN$ be equal to $2c$.

9. if MN be equal to $2c$.

10. if MN pass through the fixed point (a, b).

11. if MN be parallel to the given line $y = mx$.

12. Two fixed points A and B are taken on the axes such that $OA=a$ and $OB=b$; two variable points A' and B' are taken on the same axes; find the locus of the intersection of AB' and $A'B$

(1) when $$OA'+OB'=OA+OB,$$

and (2) when $$\frac{1}{OA'}-\frac{1}{OB'}=\frac{1}{OA}-\frac{1}{OB}.$$

13. Through a fixed point P are drawn any two straight lines to cut one fixed straight line OX in A and B and another fixed straight line OY in C and D; prove that the locus of the intersection of the straight lines AC and BD is a straight line passing through O.

14. OX and OY are two straight lines at right angles to one another; on OY is taken a fixed point A and on OX any point B; on AB an equilateral triangle is described, its vertex C being on the side of AB away from O. Shew that the locus of C is a straight line.

15. If a straight line pass through a fixed point, find the locus of the middle point of the portion of it which is intercepted between two given straight lines.

16. A and B are two fixed points; if PA and PB intersect a constant distance $2c$ from a given straight line, find the locus of P.

17. Through a fixed point O are drawn two straight lines at right angles to meet two fixed straight lines, which are also at right angles, in the points P and Q. Shew that the locus of the foot of the perpendicular from O on PQ is a straight line.

18. Find the locus of a point at which two given portions of the same straight line subtend equal angles.

19. Find the locus of a point which moves so that the difference of its distances from two fixed straight lines at right angles is equal to its distance from a fixed straight line.

20. A straight line AB, whose length is c, slides between two given oblique axes which meet at O; find the locus of the orthocentre of the triangle OAB.

21. Having given the bases and the sum of the areas of a number of triangles which have a common vertex, shew that the locus of this vertex is a straight line.

22. Through a given point O a straight line is drawn to cut two given straight lines in R and S; find the locus of a point P on this variable straight line, which is such that

(1) $2OP=OR+OS$,

and (2) $OP^2=OR \cdot OS$.

23. Given n straight lines and a fixed point O; through O is drawn a straight line meeting these lines in the points R_1, R_2, R_3, ...R_n, and on it is taken a point R such that

$$\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \frac{1}{OR_3} + \ldots + \frac{1}{OR_n};$$

shew that the locus of R is a straight line.

24. A variable straight line cuts off from n given concurrent straight lines intercepts the sum of the reciprocals of which is constant. Shew that it always passes through a fixed point.

25. If a triangle ABC remain always similar to a given triangle, and if the point A be fixed and the point B always move along a given straight line, find the locus of the point C.

26. A right-angled triangle ABC, having C a right angle, is of given magnitude, and the angular points A and B slide along two given perpendicular axes; shew that the locus of C is the pair of straight lines whose equations are $y = \pm \frac{b}{a} x$.

27. Two given straight lines meet in O, and through a given point P is drawn a straight line to meet them in Q and R; if the parallelogram $OQSR$ be completed find the equation to the locus of S.

28. Through a given point O is drawn a straight line to meet two given parallel straight lines in P and Q; through P and Q are drawn straight lines in given directions to meet in R; prove that the locus of R is a straight line.

CHAPTER VI.

ON EQUATIONS REPRESENTING TWO OR MORE STRAIGHT LINES.

106. SUPPOSE we have to trace the locus represented by the equation

$$y^2 - 3xy + 2x^2 = 0 \quad\text{.................}(1).$$

This equation is equivalent to

$$(y - x)(y - 2x) = 0 \quad\text{.................}(2).$$

It is satisfied by the coordinates of all points which make the first of these brackets equal to zero, and also by the coordinates of all points which make the second bracket zero, *i.e.* by all the points which satisfy the equation

$$y - x = 0 \quad\text{.......................}(3),$$

and also by the points which satisfy

$$y - 2x = 0 \quad\text{.......................}(4).$$

But, by Art. 47, the equation (3) represents a straight line passing through the origin, and so also does equation (4).

Hence equation (1) represents the two straight lines which pass through the origin, and are inclined at angles of 45° and $\tan^{-1} 2$ respectively to the axis of x.

107. Ex. 1. *Trace the locus* $xy = 0$. This equation is satisfied by all the points which satisfy the equation $x = 0$ and by all the points which satisfy $y = 0$, *i.e.* by all the points which lie either on the axis of y or on the axis of x.

The required locus is therefore the two axes of coordinates.

Ex. 2. *Trace the locus* $x^2 - 5x + 6 = 0$. This equation is equivalent to $(x-2)(x-3) = 0$. It is therefore satisfied by all points which satisfy the equation $x - 2 = 0$ and also by all the points which satisfy the equation $x - 3 = 0$.

But these equations represent two straight lines which are parallel to the axis of y and are at distances 2 and 3 respectively from the origin (Art. 46).

Ex. 3. *Trace the locus* $xy - 4x - 5y + 20 = 0$. This equation is equivalent to $(x-5)(y-4) = 0$, and therefore represents a straight line parallel to the axis of y at a distance 5 and also a straight line parallel to the axis of x at a distance 4.

108. Let us consider the general equation

$$ax^2 + 2hxy + by^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

On multiplying it by a it may be written in the form

$$(a^2x^2 + 2ahxy + h^2y^2) - (h^2 - ab)\,y^2 = 0,$$

i.e. $\{(ax + hy) + y\sqrt{h^2 - ab}\}\{(ax + hy) - y\sqrt{h^2 - ab}\} = 0.$

As in the last article the equation (1) therefore represents the two straight lines whose equations are

$$ax + hy + y\sqrt{h^2 - ab} = 0 \quad\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$ax + hy - y\sqrt{h^2 - ab} = 0 \quad\ldots\ldots\ldots\ldots\ldots(3),$$

each of which passes through the origin.

For (1) is satisfied by *all* the points which satisfy (2), and also by *all* the points which satisfy (3).

These two straight lines are real and different if $h^2 > ab$, real and coincident if $h^2 = ab$, and imaginary if $h^2 < ab$.

[For in the latter case the coefficient of y in each of the equations (2) and (3) is partly real and partly imaginary.]

In the case when $h^2 < ab$, the straight lines, though themselves imaginary, intersect in a real point. For the origin lies on the locus given by (1), since the equation (1) is always satisfied by the values $x = 0$ and $y = 0$.

109. An equation such as (1) of the previous article, which is such that in each term the sum of the indices of x and y is the same, is called a homogeneous equation. This equation (1) is of the second degree; for in the first term the index of x is 2; in the second term the index of both x and y is 1 and hence their sum is 2; whilst in the third term the index of y is 2.

Similarly the expression

$$3x^3 + 4x^2y - 5xy^2 + 9y^3$$

is a homogeneous expression of the third degree.

The expression

$$3x^3 + 4x^2y - 5xy^2 + 9y^3 - 7xy$$

is not however homogeneous; for in the first four terms the sum of the indices is 3 in each case, whilst in the last term this sum is 2.

From Art. 108 it follows that a homogeneous equation of the second degree represents two straight lines, real and different, coincident, or imaginary.

110. *The axes being rectangular, to find the angle between the straight lines given by the equation*

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{.................(1)}.$$

Let the separate equations to the two lines be

$$y - m_1x = 0 \text{ and } y - m_2x = 0 \qquad \text{...........(2)},$$

so that (1) must be equivalent to

$$b(y - m_1x)(y - m_2x) = 0 \qquad \text{..............(3)}.$$

Equating the coefficients of xy and x^2 in (1) and (3), we have

$$-b(m_1 + m_2) = 2h, \text{ and } bm_1m_2 = a,$$

so that $$m_1 + m_2 = -\frac{2h}{b}, \text{ and } m_1m_2 = \frac{a}{b}.$$

If θ be the angle between the straight lines (2) we have, by Art. 66,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{2\sqrt{h^2 - ab}}{a+b} \quad\text{............(4).}$$

Hence the required angle is found.

111. *Condition that the straight lines of the previous article may be (1) perpendicular, and (2) coincident.*

(1) If $a + b = 0$ the value of $\tan\theta$ is ∞ and hence θ is $90°$; the straight lines are therefore perpendicular.

Hence two straight lines, represented by one equation, are at right angles if the algebraic sum of the coefficients of x^2 and y^2 be zero.

For example, the equations

$$x^2 - y^2 = 0 \text{ and } 6x^2 + 11xy - 6y^2 = 0$$

both represent pairs of straight lines at right angles.

Similarly, whatever be the value of h, the equation

$$x^2 + 2hxy - y^2 = 0,$$

represents a pair of straight lines at right angles.

(2) If $h^2 = ab$, the value of $\tan\theta$ is zero and hence θ is zero. The angle between the straight lines is therefore zero and, since they both pass through the origin, they are therefore coincident.

This may be seen directly from the original equation. For if $h^2 = ab$, *i.e.* $h = \sqrt{ab}$, it may be written

$$ax^2 + 2\sqrt{ab}\,xy + by^2 = 0,$$

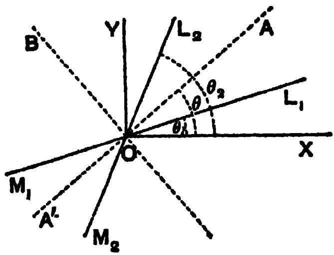
i.e.
$$(\sqrt{a}x + \sqrt{b}y)^2 = 0,$$

which is two coincident straight lines.

112. *To find the equation to the straight lines bisecting the angle between the straight lines given by*

$$ax^2 + 2hxy + by^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the equation (1) represent the two straight lines



L_1OM_1 and L_2OM_2 inclined at angles θ_1 and θ_2 to the axis of x, so that (1) is equivalent to

$$b(y - x\tan\theta_1)(y - x\tan\theta_2) = 0.$$

Hence

$$\tan\theta_1 + \tan\theta_2 = -\frac{2h}{b}, \text{ and } \tan\theta_1\tan\theta_2 = \frac{a}{b}\ldots(2).$$

Let OA and OB be the required bisectors.

Since $\qquad \angle AOL_1 = \angle L_2OA,$

$$\therefore\ \angle AOX - \theta_1 = \theta_2 - \angle AOX.$$

$$\therefore\ 2\angle AOX = \theta_1 + \theta_2.$$

Also $\qquad \angle BOX = 90° + \angle AOX.$

$$\therefore\ 2\angle BOX = 180° + \theta_1 + \theta_2.$$

Hence, if θ stand for *either* of the angles AOX or BOX, we have

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = -\frac{2h}{b-a},$$

by equations (2).

But, if (x, y) be the coordinates of any point on either of the lines OA or OB, we have

$$\tan\theta = \frac{y}{x}.$$

$$\therefore -\frac{2h}{b-a} = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\dfrac{y}{x}}{1-\dfrac{y^2}{x^2}} = \frac{2xy}{x^2-y^2},$$

$$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}} = \frac{\mathbf{xy}}{\mathbf{h}}.$$

This, being a relation holding between the coordinates of *any* point on *either* of the bisectors, is, by Art. 42, the equation to the bisectors.

118. The foregoing equation may also be obtained in the following manner:

Let the given equation represent the straight lines

$$y - m_1 x = 0 \text{ and } y - m_2 x = 0 \text{.................... (1)},$$

so that
$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b} \text{ .... (2)}.$$

The equations to the bisectors of the angles between the straight lines (1) are, by Art. 84,

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \frac{y-m_2x}{\sqrt{1+m_2^2}} \text{ and } \frac{y-m_1x}{\sqrt{1+m_1^2}} = -\frac{y-m_2x}{\sqrt{1+m_2^2}},$$

or, expressed in one equation,

$$\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}} - \frac{y-m_2x}{\sqrt{1+m_2^2}}\right\}\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}} + \frac{y-m_2x}{\sqrt{1+m_2^2}}\right\} = 0,$$

i.e.
$$\frac{(y-m_1x)^2}{1+m_1^2} - \frac{(y-m_2x)^2}{1+m_2^2} = 0,$$

i.e. $(1+m_2^2)(y^2-2m_1xy+m_1^2x^2) - (1+m_1^2)(y^2-2m_2xy+m_2^2x^2) = 0,$

i.e.
$$(m_1^2-m_2^2)(x^2-y^2) + 2(m_1m_2-1)(m_1-m_2)xy = 0,$$

i.e.
$$(m_1+m_2)(x^2-y^2) + 2(m_1m_2-1)xy = 0.$$

Hence, by (2), the required equation is

$$\frac{-2h}{b}(x^2-y^2) + 2\left(\frac{a}{b}-1\right)xy = 0,$$

i.e.
$$\frac{x^2-y^2}{a-b} = \frac{xy}{h}.$$

EXAMPLES. XII.

Find what straight lines are represented by the following equations and determine the angles between them.

1. $x^2-7xy+12y^2=0.$
2. $4x^2-24xy+11y^2=0.$
3. $33x^2-71xy-14y^2=0.$
4. $x^3-6x^2+11x-6=0.$
5. $y^2-16=0.$
6. $y^3-xy^2-14x^2y+24x^3=0.$
7. $x^2+2xy\sec\theta+y^2=0.$
8. $x^2+2xy\cot\theta+y^2=0.$

9. Find the equations of the straight lines bisecting the angles between the pairs of straight lines given in examples 2, 3, 7, and 8.

10. Shew that the two straight lines

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$$

make with the axis of x angles such that the difference of their tangents is 2.

11. Prove that the two straight lines

$$(x^2+y^2)(\cos^2\theta\sin^2\alpha+\sin^2\theta)=(x\tan\alpha-y\sin\theta)^2$$

include an angle 2α.

12. Prove that the two straight lines

$$x^2\sin^2\alpha\cos^2\theta+4xy\sin\alpha\sin\theta+y^2[4\cos\alpha-(1+\cos\alpha)^2\cos^2\theta]=0$$

meet at an angle α.

GENERAL EQUATION OF THE SECOND DEGREE.

114. The most general expression, which contains terms involving x and y in a degree not higher than the second, must contain terms involving x^2, xy, y^2, x, y, and a constant.

The notation which is in general use for this expression is

$$ax^2+2hxy+by^2+2gx+2fy+c \,\ldots\ldots\ldots\, (1).$$

The quantity (1) is known as the general expression of the second degree, and when equated to zero is called the **general equation of the second degree.**

The student may better remember the seemingly arbitrary coefficients of the terms in the expression (1) if the reason for their use be given.

The most general expression involving terms only of the second degree in x, y, and z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \text{ ...... } (2),$$

where the coefficients occur in the order of the alphabet.

If in this expression we put z equal to unity we get

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy,$$

which, after rearrangement, is the same as (1).

Now in Solid Geometry we use three coordinates x, y, and z. Also many formulæ in Plane Geometry are derived from those of Solid Geometry by putting z equal to unity.

We therefore, in Plane Geometry, use that notation corresponding to which we have the standard notation in Solid Geometry.

115. In general, as will be shewn in Chapter XV., the general equation represents a Curve-Locus.

If a certain condition holds between the coefficients of its terms it will, however, represent a pair of straight lines.

This condition we shall determine in the following article.

116. *To find the condition that the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ ....} (1)$$

may represent two straight lines.

If we can break the left-hand members of (1) into two factors, each of the first degree, then, as in Art. 108, it will represent two straight lines.

If a be not zero, multiply equation (1) by a and arrange in powers of x; it then becomes

$$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac.$$

On completing the square on the left hand we have

$$a^2x^2 + 2ax(hy + g) + (hy + g)^2 = y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac,$$

i.e.

$$(ax + hy + g) = \pm\sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac} \text{ ...} (2).$$

From (2) we cannot obtain x in terms of y, involving only terms of the *first* degree, unless the quantity under the radical sign be a perfect square.

The condition for this is

$$(gh - af)^2 = (h^2 - ab)(g^2 - ac),$$

i.e. $$g^2h^2 - 2afgh + a^2f^2 = g^2h^2 - abg^2 - ach^2 + a^2bc.$$

Cancelling and dividing by a, we have the required condition, viz.

$$\mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0} \quad \text{......(3)}.$$

117. The foregoing condition may be otherwise obtained thus: The given equation, multiplied by (a), is

$$a^2x^2 + 2ahxy + aby^2 + 2agx + 2afy + ac = 0 \quad \text{............(4)}.$$

The terms of the second degree in this equation break up, as in Art. 108, into the factors

$$ax + hy - y\sqrt{h^2 - ab} \text{ and } ax + hy + y\sqrt{h^2 - ab}.$$

If then (4) break into factors it must be equivalent to

$$\{ax + (h - \sqrt{h^2 - ab})\,y + A\}\,\{ax + (h + \sqrt{h^2 - ab})\,y + B\} = 0,$$

where A and B are given by the relations

$$a(A + B) = 2ga \quad \text{......... .... (5)},$$

$$A(h + \sqrt{h^2 - ab}) + B(h - \sqrt{h^2 - ab}) = 2fa \quad \text{............(6)},$$

and $$AB = ac \quad \text{................................ (7)}.$$

The equations (5) and (6) give

$$A + B = 2g, \text{ and } A - B = \frac{2fa - 2gh}{\sqrt{h^2 - ab}}.$$

The relation (7) then gives

$$4ac = 4AB = (A + B)^2 - (A - B)^2$$

$$= 4g^2 - 4\frac{(fa - gh)^2}{h^2 - ab},$$

i.e. $$(fa - gh)^2 = (g^2 - ac)(h^2 - ab),$$

which, as before, reduces to

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Ex. If a be zero, prove that the general equation will represent two straight lines if

$$2fgh - bg^2 - ch^2 = 0.$$

If both a and b be zero, prove that the condition is $2fg - ch = 0$.

118. The relation (3) of Art. 116 is equivalent to the expression

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

This may be easily verified by writing down the value of the determinant by the rule of Art. 5.

A geometrical meaning to this form of the relation (3) will be given in a later chapter. [Art. 355.]

The quantity on the left-hand side of equation (3) is called the **Discriminant** of the General Equation.

The general equation therefore represents two straight lines if its discriminant be zero.

119. Ex. 1. *Prove that the equation*

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

represents two straight lines, and find the angle between them.

Here

$$a = 12, \quad h = \tfrac{7}{2}, \quad b = -10, \quad g = \tfrac{13}{2}, \quad f = \tfrac{45}{2}, \text{ and } c = -35.$$

Hence $\qquad abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 12 \times (-10) \times (-35) + 2 \times \tfrac{45}{2} \times \tfrac{13}{2} \times \tfrac{7}{2} - 12 \times (\tfrac{45}{2})^2 - (-10) \times (\tfrac{13}{2})^2 - (-35)(\tfrac{7}{2})^2$$

$$= 4200 + \tfrac{4095}{4} - 6075 + \tfrac{1690}{4} + \tfrac{1715}{4}$$

$$= -1875 + \tfrac{7500}{4} = 0.$$

The equation therefore represents two straight lines.

Solving it for x, we have

$$x^2 + x\frac{7y+13}{12} + \left(\frac{7y+13}{24}\right)^2 = \frac{10y^2 - 45y + 35}{12} + \left(\frac{7y+13}{24}\right)^2$$

$$= \left(\frac{23y-43}{24}\right)^2.$$

$$\therefore \quad x + \frac{7y+13}{24} = \pm \frac{23y-43}{24},$$

i. e. $\qquad x = \dfrac{2y-7}{3}$ or $\dfrac{-5y+5}{4}$.

The given equation therefore represents the two straight lines

$$3x = 2y - 7 \text{ and } 4x = -5y + 5.$$

The "m's" of these two lines are therefore $\frac{3}{2}$ and $-\frac{4}{5}$, and the angle between them, by Art 66,

$$=\tan^{-1}\frac{\frac{3}{2}-\left(-\frac{4}{5}\right)}{1+\frac{3}{2}\left(-\frac{4}{5}\right)}=\tan^{-1}\left(-\tfrac{23}{2}\right).$$

Ex. 2. *Find the value of h so that the equation*

$$6x^2+2hxy+12y^2+22x+31y+20=0$$

may represent two straight lines.

Here

$$a=6,\quad b=12,\quad g=11,\quad f=\tfrac{31}{2},\text{ and } c=20.$$

The condition (3) of Art. 116 then gives

$$20h^2-341h+\tfrac{2907}{2}=0,$$

i.e.
$$\left(h-\tfrac{17}{2}\right)(20h-171)=0.$$

Hence
$$h=\tfrac{17}{2}\text{ or }\tfrac{171}{20}.$$

Taking the first of these values, the given equation becomes

$$6x^2+17xy+12y^2+22x+31y+20=0,$$

i.e.
$$(2x+3y+4)(3x+4y+5)=0$$

Taking the second value, the equation is

$$20x^2+57xy+40y^2+\tfrac{220}{3}x+\tfrac{310}{3}y+\tfrac{200}{3}=0,$$

i.e.
$$\left(4x+5y+\tfrac{20}{3}\right)(5x+8y+10)=0.$$

EXAMPLES. XIII.

Prove that the following equations represent two straight lines; find also their point of intersection and the angle between them.

1. $6y^2-xy-x^2+30y+36=0$. 2. $x^2-5xy+4y^2+x+2y-2=0$.

3. $3y^2-8xy-3x^2-29x+3y-18=0$.

4. $y^2+xy-2x^2-5x-y-2=0$.

5. Prove that the equation

$$x^2+6xy+9y^2+4x+12y-5=0$$

represents two parallel lines.

Find the value of k so that the following equations may represent pairs of straight lines:

6. $6x^2+11xy-10y^2+x+31y+k=0$.

7. $12x^2-10xy+2y^2+11x-5y+k=0$.

8. $12x^2+kxy+2y^2+11x-5y+2=0$.

9. $6x^2+xy+ky^2-11x+43y-35=0$.

10. $kxy - 8x + 9y - 12 = 0.$

11. $x^2 + \frac{10}{3}xy + y^2 - 5x - 7y + k = 0.$

12. $12x^2 + xy - 6y^2 - 29x + 8y + k = 0.$

13. $2x^2 + xy - y^2 + kx + 6y - 9 = 0.$

14. $x^2 + kxy + y^2 - 5x - 7y + 6 = 0.$

15. Prove that the equations to the straight lines passing through the origin which make an angle a with the straight line $y + x = 0$ are given by the equation

$$x^2 + 2xy \sec 2a + y^2 = 0.$$

16. What relations must hold between the coefficients of the equations

(i) $ax^2 + by^2 + cx + cy = 0,$

and (ii) $ay^2 + bxy + dy + ex = 0,$

so that each of them may represent a pair of straight lines?

17. The equations to a pair of opposite sides of a parallelogram are

$$x^2 - 7x + 6 = 0 \text{ and } y^2 - 14y + 40 = 0;$$

find the equations to its diagonals.

120. *To prove that a homogeneous equation of the nth degree represents n straight lines, real or imaginary, which all pass through the origin.*

Let the equation be

$$y^n + A_1xy^{n-1} + A_2x^2y^{n-2} + A_3x^3y^{n-3} + \dots + A_nx^n = 0.$$

On division by x^n, it may be written

$$\left(\frac{y}{x}\right)^n + A_1\left(\frac{y}{x}\right)^{n-1} + A_2\left(\frac{y}{x}\right)^{n-2} + \dots + A_n = 0 \quad \dots (1).$$

This is an equation of the nth degree in $\frac{y}{x}$, and hence must have n roots.

Let these roots be $m_1, m_2, m_3, \dots m_n$. Then (C. Smith's Algebra, Art. 89) the equation (1) must be equivalent to the equation

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_3\right) \dots \left(\frac{y}{x} - m_n\right) = 0 \dots (2).$$

The equation (2) is satisfied by *all* the points which satisfy the separate equations

$$\frac{y}{x} - m_1 = 0, \ \frac{y}{x} - m_2 = 0, \dots \frac{y}{x} - m_n = 0,$$

i.e. by all the points which lie on the n straight lines

$$y - m_1 x = 0, \quad y - m_2 x = 0, \dots y - m_n x = 0,$$

all of which pass through the origin. Conversely, the coordinates of all the points which satisfy these n equations satisfy equation (1). Hence the proposition.

121. Ex. 1. The equation

$$y^3 - 6xy^2 + 11x^2 y - 6x^3 = 0,$$

which is equivalent to

$$(y - x)(y - 2x)(y - 3x) = 0,$$

represents the three straight lines

$$y - x = 0, \quad y - 2x = 0, \quad \text{and} \quad y - 3x = 0,$$

all of which pass through the origin.

Ex. 2. The equation $y^3 - 5y^2 + 6y = 0$,

i.e. $$y(y-2)(y-3) = 0,$$

similarly represents the three straight lines

$$y = 0, \quad y = 2, \quad \text{and} \quad y = 3,$$

all of which are parallel to the axis of x.

122. *To find the equation to the two straight lines joining the origin to the points in which the straight line*

$$lx + my = n \qquad\qquad (1)$$

meets the locus whose equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad\qquad (2).$$

The equation (1) may be written

$$\frac{lx + my}{n} = 1 \qquad\qquad (3).$$

The coordinates of the points in which the straight line meets the locus satisfy both equation (2) and equation (3), and hence satisfy the equation

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\frac{lx + my}{n} + c\left(\frac{lx + my}{n}\right)^2 = 0 \qquad\qquad (4).$$

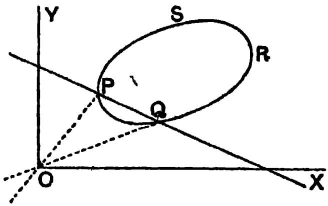
[For at the points where (3) and (4) are true it is clear that (2) is true.]

Hence (4) represents *some locus* which passes through the intersections of (2) and (3).

But, since the equation (4) is homogeneous and of the second degree, it represents two straight lines passing through the origin (Art. 108).

It therefore must represent the two straight lines joining the origin to the intersections of (2) and (3).

123. The preceding article may be illustrated geometrically if we assume that the equation (2) represents some such curve as $PQRS$ in the figure.



Let the given straight line cut the curve in the points P and Q.

The equation (2) holds for all points on the curve $PQRS$.

The equation (3) holds for all points on the line PQ.

Both equations are therefore true at the points of intersection P and Q.

The equation (4), which is derived from (2) and (3), holds therefore at P and Q.

But the equation (4) represents two straight lines, each of which passes through the point O.

It must therefore represent the two straight lines OP and OQ.

124. Ex. *Prove that the straight lines joining the origin to the points of intersection of the straight line $x-y=2$ and the curve*

$$5x^2+12xy-8y^2+8x-4y+12=0$$

make equal angles with the axes.

As in Art. 122 the equation to the required straight lines is

$$5x^2+12xy-8y^2+(8x-4y)\frac{x-y}{2}+12\left(\frac{x-y}{2}\right)^2=0 \quad\text{......(1).}$$

For this equation is homogeneous and therefore represents two straight lines through the origin; also it is satisfied at the points where the two given equations are satisfied.

Now (1) is, on reduction,

$$y^2 = 4x^2,$$

so that the equations to the two lines are

$$y = 2x \text{ and } y = -2x.$$

These lines are equally inclined to the axes.

125. It was stated in Art. 115 that, *in general,* an equation of the second degree represents a curve-line, including (Art. 116) as a particular case two straight lines.

In some cases however it will be found that such equations only represent isolated points. Some examples are appended.

Ex. 1. *What is represented by the locus*

$$(x - y + c)^2 + (x + y - c)^2 = 0? \qquad \text{.........(1).}$$

We know that the sum of the squares of two real quantities cannot be zero unless each of the squares is separately zero.

The only real points that satisfy the equation (1) therefore satisfy both of the equations

$$x - y + c = 0 \text{ and } x + y - c = 0.$$

But the only solution of these two equations is

$$x = 0, \text{ and } y = c.$$

The only real point represented by equation (1) is therefore $(0, c)$.

The same result may be obtained in a different manner. The equation (1) gives

$$(x - y + c)^2 = -(x + y - c)^2,$$

i.e. $$x - y + c = \pm\sqrt{-1}\,(x + y - c).$$

It therefore represents the two imaginary straight lines

$$x(1 - \sqrt{-1}) - y(1 + \sqrt{-1}) + c(1 + \sqrt{-1}) = 0,$$

and $$x(1 + \sqrt{-1}) - y(1 - \sqrt{-1}) + c(1 - \sqrt{-1}) = 0.$$

Each of these two straight lines passes through the real point $(0, c)$. We may therefore say that (1) represents two imaginary straight lines passing through the point $(0, c)$.

Ex. 2. *What is represented by the equation*

$$(x^2 - a^2)^2 + (y^2 - b^2)^2 = 0?$$

As in the last example, the only real points on the locus are those that satisfy *both* of the equations

$$x^2 - a^2 = 0 \text{ and } y^2 - b^2 = 0,$$

i.e. $$x = \pm a, \text{ and } y = \pm b.$$

The points represented are therefore

$$(a, b),\ (a, -b),\ (-a, b), \text{ and } (-a, -b).$$

Ex. 3. *What is represented by the equation*

$$x^2 + y^2 + a^2 = 0?$$

The only real points on the locus are those that satisfy all three of the equations

$$x = 0,\ y = 0, \text{ and } a = 0.$$

Hence, unless a vanishes, there are no such points, and the given equation represents nothing real.

The equation may be written

$$x^2 + y^2 = -a^2,$$

so that it represents points whose distance from the origin is $a\sqrt{-1}$. It therefore represents the *imaginary* circle whose radius is $a\sqrt{-1}$ and whose centre is the origin.

126. Ex. 1. *Obtain the condition that one of the straight lines given by the equation*

$$ax^2 + 2hxy + by^2 = 0 \text{ ....} (1)$$

may coincide with one of those given by the equation

$$a'x^2 + 2h'xy + b'y^2 = 0 \text{ ....} (2).$$

Let the equation to the common straight line be

$$y - m_1x = 0 \text{ ....} (3).$$

The quantity $y - m_1x$ must therefore be a factor of the left-hand of both (1) and (2), and therefore the value $y = m_1x$ must satisfy both (1) and (2).

We therefore have

$$bm_1^2+2hm_1+a=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and

$$b'm_1^2+2h'm_1+a'=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Solving (4) and (5), we have

$$\frac{m_1^2}{2(ha'-h'a)}=\frac{m_1}{ab'-a'b}=\frac{1}{2(bh'-b'h)}.$$

$$\therefore \frac{ha'-h'a}{bh'-b'h}=m_1^2=\left\{\frac{ab'-a'b}{2(bh'-b'h)}\right\}^2,$$

so that we must have

$$(ab'-a'b)^2=4(ha'-h'a)(bh'-b'h).$$

Ex. 2. *Prove that the equation*

$$m(x^3-3xy^2)+y^3-3x^2y=0$$

represents three straight lines equally inclined to one another.

Transforming to polar coordinates (Art. 35) the equation gives

$$m(\cos^3\theta-3\cos\theta\sin^2\theta)+\sin^3\theta-3\cos^2\theta\sin\theta=0,$$

i.e. $$m(1-3\tan^2\theta)+\tan^3\theta-3\tan\theta=0,$$

i.e. $$m=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\tan 3\theta.$$

If $m=\tan\alpha$, this equation gives

$$\tan 3\theta=\tan\alpha,$$

the solutions of which are

$$3\theta=\alpha, \text{ or } 180^\circ+\alpha, \text{ or } 360^\circ+\alpha,$$

i.e. $$\theta=\frac{\alpha}{3}, \text{ or } 60^\circ+\frac{\alpha}{3}, \text{ or } 120^\circ+\frac{\alpha}{3}.$$

The locus is therefore three straight lines through the origin inclined at angles

$$\frac{\alpha}{3},\quad 60^\circ+\frac{\alpha}{3},\quad\text{and}\quad 120^\circ+\frac{\alpha}{3}$$

to the axis of x.

They are therefore equally inclined to one another.

Ex. 3. *Prove that two of the straight lines represented by the equation*

$$ax^3+bx^2y+cxy^2+dy^3=0\quad\ldots\ldots\ldots\ldots\ldots(1)$$

will be at right angles if

$$a^2+ac+bd+d^2=0.$$

Let the separate equations to the three lines be

$$y-m_1x=0,\quad y-m_2x=0,\quad\text{and}\quad y-m_3x=0,$$

so that the equation (1) must be equivalent to

$$d(y-m_1x)(y-m_2x)(y-m_3x)=0,$$

and therefore $$m_1+m_2+m_3=-\frac{c}{d}\quad\text{.........................(2)},$$

$$m_2m_3+m_3m_1+m_1m_2=\frac{b}{d}\quad\text{.....................(3)},$$

and $$m_1m_2m_3=-\frac{a}{d}\quad\text{............................(4)}.$$

If the first two of these straight lines be at right angles we have, in addition,

$$m_1m_2=-1\quad\text{.............................(5)}.$$

From (4) and (5), we have

$$m_3=\frac{a}{d},$$

and therefore, from (2),

$$m_1+m_2=-\frac{c}{d}-\frac{a}{d}=-\frac{c+a}{d}.$$

The equation (3) then becomes

$$\frac{a}{d}\left(-\frac{c+a}{d}\right)-1=\frac{b}{d},$$

i.e. $$a^2+ac+bd+d^2=0.$$

EXAMPLES. XIV.

1. Prove that the equation

$$y^3-x^3+3xy(y-x)=0$$

represents three straight lines equally inclined to one another.

2. Prove that the equation

$$y^2(\cos\alpha+\sqrt{3}\sin\alpha)\cos\alpha-xy(\sin 2\alpha-\sqrt{3}\cos 2\alpha)$$
$$+x^2(\sin\alpha-\sqrt{3}\cos\alpha)\sin\alpha=0$$

represents two straight lines inclined at 60° to each other.

Prove also that the area of the triangle formed with them by the straight line

$$(\cos\alpha-\sqrt{3}\sin\alpha)y-(\sin\alpha+\sqrt{3}\cos\alpha)x+a=0$$

is $$\frac{a^2}{4\sqrt{3}},$$

and that this triangle is equilateral.

3. Shew that the straight lines

$$(A^2-3B^2)x^2+8ABxy+(B^2-3A^2)y^2=0$$

form with the line $Ax+By+C=0$ an equilateral triangle whose area is $\dfrac{C^2}{\sqrt{3}(A^2+B^2)}$.

4. Find the equation to the pair of straight lines joining the origin to the intersections of the straight line $y=mx+c$ and the curve

$$x^2+y^2=a^2.$$

Prove that they are at right angles if

$$2c^2=a^2(1+m^2).$$

5. Prove that the straight lines joining the origin to the points of intersection of the straight line

$$kx+hy=2hk$$

with the curve $\qquad (x-h)^2+(y-k)^2=c^2$

are at right angles if $\qquad h^2+k^2=c^2.$

6. Prove that the angle between the straight lines joining the origin to the intersection of the straight line $y=3x+2$ with the curve

$$x^2+2xy+3y^2+4x+8y-11=0 \text{ is } \tan^{-1}\frac{2\sqrt{2}}{3}.$$

7. Shew that the straight lines joining the origin to the other two points of intersection of the curves whose equations are

$$ax^2+2hxy+by^2+2gx=0$$

and $\qquad a'x^2+2h'xy+b'y^2+2g'x=0$

will be at right angles if

$$g(a'+b')-g'(a+b)=0.$$

What loci are represented by the equations

8. $x^2-y^2=0.$ 9. $x^2-xy=0.$ 10. $xy-ay=0.$

11. $x^3-x^2-x+1=0.$ 12. $x^3-xy^2=0.$ 13. $x^3+y^3=0.$

14. $x^2+y^2=0.$ 15. $x^2y=0.$ 16. $(x^2-1)(y^2-4)=0.$

17. $(x^2-1)^2+(y^2-4)^2=0.$ 18. $(y-mx-c)^2+(y-m'x-c')^2=0.$

19. $(x^2-a^2)^2(x^2-b^2)^2+c^4(y^2-a^2)^2=0.$ 20. $(x-a)^2-y^2=0.$

21. $(x+y)^2-c^2=0.$ 22. $r=a\sec(\theta-\alpha).$

23. Shew that the equation

$$bx^2-2hxy+ay^2=0$$

represents a pair of straight lines which are at right angles to the pair given by the equation

$$ax^2+2hxy+by^2=0.$$

24. If pairs of straight lines

$$x^2-2pxy-y^2=0 \text{ and } x^2-2qxy-y^2=0$$

be such that each pair bisects the angles between the other pair, prove that $pq=-1$.

25. Prove that the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0$$

is equally inclined to the pair

$$ax^2+2hxy+by^2=0.$$

26. Shew also that the pair

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$$

is equally inclined to the same pair.

27. If one of the straight lines given by the equation

$$ax^2+2hxy+by^2=0$$

coincide with one of those given by

$$a'x^2+2h'xy+b'y^2=0,$$

and the other lines represented by them be perpendicular, prove that

$$\frac{ha'b'}{b'-a'}=\frac{h'ab}{b-a}=\tfrac{1}{2}\sqrt{-aa'bb'}.$$

28. Prove that the equation to the bisectors of the angle between the straight lines $ax^2+2hxy+by^2=0$ is

$$h(x^2-y^2)+(b-a)xy=(ax^2-by^2)\cos\omega,$$

the axes being inclined at an angle ω.

29. Prove that the straight lines

$$ax^2+2hxy+by^2=0$$

make equal angles with the axis of x if $h=a\cos\omega$, the axes being inclined at an angle ω.

30. If the axes be inclined at an angle ω, shew that the equation

$$x^2+2xy\cos\omega+y^2\cos 2\omega=0$$

represents a pair of perpendicular straight lines.

31. Shew that the equation

$$\cos 3\alpha\,(x^3-3xy^2)+\sin 3\alpha\,(y^3-3x^2y)+3a(x^2+y^2)-4a^3=0$$

represents three straight lines forming an equilateral triangle.

Prove also that its area is $3\sqrt{3}a^2$.

32. Prove that the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel straight lines if

$$h^2=ab \text{ and } bg^2=af^2.$$

Prove also that the distance between them is

$$2\sqrt{\frac{g^2-ac}{a(a+b)}}.$$

33. If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represent a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points where these meet the axes is

$$ax^2-2hxy+by^2+2gx+2fy+c+\frac{4fg}{c}xy=0.$$

34. If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represent two straight lines, prove that the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2}.$$

35. Shew that the orthocentre of the triangle formed by the straight lines

$$ax^2+2hxy+by^2=0 \text{ and } lx+my=1$$

is a point (x', y') such that

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$

36. Hence find the locus of the orthocentre of a triangle of which two sides are given in position and whose third side goes through a fixed point.

37. Shew that the distance between the points of intersection of the straight line

$$x\cos\alpha+y\sin\alpha-p=0$$

with the straight lines $\quad ax^2+2hxy+by^2=0$

is
$$\frac{2p\sqrt{h^2-ab}}{b\cos^2\alpha-2h\cos\alpha\sin\alpha+a\sin^2\alpha}.$$

Deduce the area of the triangle formed by them.

38. Prove that the product of the perpendiculars let fall from the point (x', y') upon the pair of straight lines

$$ax^2+2hxy+by^2=0$$

is
$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

39. Shew that two of the straight lines represented by the equation

$$ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$$

will be at right angles if

$$(b+d)(ad+be)+(e-a)^2(a+c+e)=0.$$

40. Prove that two of the lines represented by the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4=0$$

will bisect the angles between the other two if

$$c+6a=0 \text{ and } b+d=0.$$

41. Prove that one of the lines represented by the equation

$$ax^3+bx^2y+cxy^2+dy^3=0$$

will bisect the angle between the other two if

$$(3a+c)^2(bc+2cd-3ad)=(b+3d)^2(bc+2ab-3ad).$$

CHAPTER VII.

TRANSFORMATION OF COORDINATES.

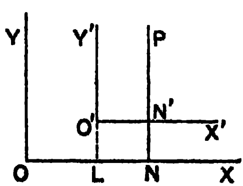
127. It is sometimes found desirable in the discussion of problems to alter the origin and axes of coordinates, either by altering the origin without alteration of the direction of the axes, or by altering the directions of the axes and keeping the origin unchanged, or by altering the origin and also the directions of the axes. The latter case is merely a combination of the first two. Either of these processes is called a transformation of coordinates.

We proceed to establish the fundamental formulæ for such transformation of coordinates.

128. *To alter the origin of coordinates without altering the directions of the axes.*

Let OX and OY be the original axes and let the new axes, parallel to the original, be

$$O'X' \text{ and } O'Y'.$$



Let the coordinates of the new origin O', referred to the original axes be h and k, so that, if $O'L$ be perpendicular to OX, we have

$$OL = h \text{ and } LO' = k.$$

Let P be any point in the plane of the paper, and let its coordinates, referred to the original axes, be x and y, and referred to the new axes let them be x' and y'.

Draw PN perpendicular to OX to meet $O'X'$ in N'.

Then

$$ON = x, \quad NP = y, \quad O'N' = x', \text{ and } N'P = y'.$$

We therefore have

$$x = ON = OL + O'N' = h + x',$$

and $$y = NP = LO' + N'P = k + y'.$$

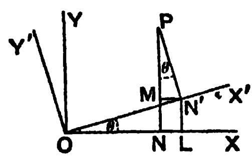
The origin is therefore transferred to the point (h, k) when we substitute for the coordinates x and y the quantities

$$x' + h \text{ and } y' + k.$$

The above article is true whether the axes be oblique or rectangular.

129. *To change the direction of the axes of coordinates, without changing the origin, both systems of coordinates being rectangular.*

Let OX and OY be the original system of axes and OX' and OY' the new system, and let the angle, XOX', through which the axes are turned be called θ.



Take any point P in the plane of the paper.

Draw PN and PN' perpendicular to OX and OX', and also $N'L$ and $N'M$ perpendicular to OX and PN.

If the coordinates of P, referred to the original axes, be x and y, and, referred to the new axes, be x' and y', we have

$$ON = x, \quad NP = y, \quad ON' = x', \text{ and } N'P = y'.$$

The angle

$$MPN' = 90° - \angle MN'P = \angle MN'O = \angle XOX' = \theta.$$

We then have

$$x = ON = OL - MN' = ON' \cos\theta - N'P \sin\theta$$
$$= x' \cos\theta - y' \sin\theta \quad \text{.......................} \quad (1),$$

and $$y = NP = LN' + MP = ON' \sin\theta + N'P \cos\theta$$
$$= x' \sin\theta + y' \cos\theta \quad \text{.......................} \quad (2).$$

If therefore in any equation we wish to turn the axes, being rectangular, through an angle θ we must substitute

$$\mathbf{x' \cos\theta - y' \sin\theta} \text{ and } \mathbf{x' \sin\theta + y' \cos\theta}$$

for x and y.

When we have both to change the origin, and also the direction of the axes, the transformation is clearly obtained by combining the results of the previous articles.

If the origin is to be transformed to the point (h, k) and the axes to be turned through an angle θ, we have to substitute

$$h + x' \cos\theta - y' \sin\theta \text{ and } k + x' \sin\theta + y' \cos\theta$$

for x and y respectively.

The student, who is acquainted with the theory of projection of straight lines, will see that equations (1) and (2) express the fact that the projections of OP on OX and OY are respectively equal to the sum of the projections of ON' and $N'P$ on the same two lines.

130. Ex. 1. *Transform to parallel axes through the point* $(-2, 3)$ *the equation*

$$2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0.$$

We substitute $x = x' - 2$ and $y = y' + 3$, and the equation becomes

$$2(x'-2)^2 + 4(x'-2)(y'+3) + 5(y'+3)^2 - 4(x'-2) - 22(y'+3) + 7 = 0,$$

i.e. $$2x'^2 + 4x'y' + 5y'^2 - 22 = 0.$$

Ex. 2. *Transform to axes inclined at* 30° *to the original axes the equation*

$$x^2 + 2\sqrt{3}xy - y^2 = 2a^2.$$

For x and y we have to substitute

$$x' \cos 30° - y' \sin 30° \text{ and } x' \sin 30° + y' \cos 30°,$$

i.e. $$\frac{x'\sqrt{3} - y'}{2} \text{ and } \frac{x' + y'\sqrt{3}}{2}.$$

The equation then becomes

$$(x'\sqrt{3} - y')^2 + 2\sqrt{3}(x'\sqrt{3} - y')(x' + y'\sqrt{3}) - (x' + y'\sqrt{3})^2 = 8a^2,$$

i.e. $$x'^2 - y'^2 = a^2.$$

EXAMPLES. XV.

1. Transform to parallel axes through the point $(1, -2)$ the equations

(1) $y^2-4x+4y+8=0$,

and (2) $2x^2+y^2-4x+4y=0$.

2. What does the equation

$$(x-a)^2+(y-b)^2=c^2$$

become when it is transferred to parallel axes through

(1) the point $(a-c, b)$,

(2) the point $(a, b-c)$?

3. What does the equation

$$(a-b)(x^2+y^2)-2abx=0$$

become if the origin be moved to the point $\left(\frac{ab}{a-b}, 0\right)$?

4. Transform to axes inclined at 45° to the original axes the equations

(1) $x^2-y^2=a^2$,

(2) $17x^2-16xy+17y^2=225$,

and (3) $y^4+x^4+6x^2y^2=2$.

5. Transform to axes inclined at an angle α to the original axes the equations

(1) $x^2+y^2=r^2$,

and (2) $x^2+2xy\tan 2\alpha-y^2=a^2$.

6. If the axes be turned through an angle $\tan^{-1} 2$, what does the equation $4xy-3x^2=a^2$ become?

7. By transforming to parallel axes through a properly chosen point (h, k), prove that the equation

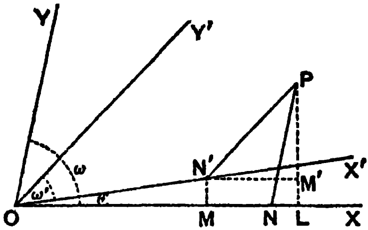
$$12x^2-10xy+2y^2+11x-5y+2=0$$

can be reduced to one containing only terms of the second degree.

8. Find the angle through which the axes may be turned so that the equation $$Ax+By+C=0$$ may be reduced to the form $x=$constant, and determine the value of this constant.

131. The general proposition, which is given in the next article, on the transformation from one set of oblique axes to any other set of oblique axes is of very little importance and is hardly ever required.

***132.** *To change from one set of axes, inclined at an angle ω, to another set, inclined at an angle ω', the origin remaining unaltered.*



Let OX and OY be the original axes, OX' and OY' the new axes, and let the angle XOX' be θ.

Take any point P in the plane of the paper.

Draw PN and PN' parallel to OY and OY' to meet OX and OX' respectively in N and N', PL perpendicular to OX, and $N'M$ and $N'M'$ perpendicular to OL and LP.

Now

$$\angle PNL = \angle YOX = \omega, \text{ and } PN'M' = Y'OX = \omega' + \theta.$$

Hence if

$$ON = x, \quad NP = y, \quad ON' = x', \text{ and } N'P = y',$$

we have $\qquad y \sin \omega = NP \sin \omega = LP = MN' + M'P$

$$= ON' \sin \theta + N'P \sin (\omega' + \theta),$$

so that $\qquad y \sin \omega = x' \sin \theta + y' \sin (\omega' + \theta) \quad\text{..............(1)}.$

Also

$$x + y \cos \omega = ON + NL = OL = OM + N'M'$$
$$= x' \cos \theta + y' \cos (\omega' + \theta) \quad\text{..............(2)}.$$

Multiplying (2) by $\sin \omega$, (1) by $\cos \omega$, and subtracting, we have

$$x \sin \omega = x' \sin (\omega - \theta) + y' \sin (\omega - \omega' - \theta) \quad\text{......(3)}.$$

[This equation (3) may also be obtained by drawing a perpendicular from P upon OY and proceeding as for equation (1).]

The equations (1) and (3) give the proper substitutions for the change of axes in the general case.

As in Art. 130 the equations (1) and (2) may be obtained by equating the projections of OP and of ON' and $N'P$ on OX and a straight line perpendicular to OX.

***133.** *Particular cases of the preceding article.*

(1) Suppose we wish to transfer our axes from a rectangular pair to one inclined at an angle ω'. In this case ω is 90°, and the formulæ of the preceding article become

$$x = x' \cos\theta + y' \cos(\omega' + \theta),$$

and
$$y = x' \sin\theta + y' \sin(\omega' + \theta).$$

(2) Suppose the transference is to be from oblique axes, inclined at ω, to rectangular axes. In this case ω' is 90°, and our formulæ become

$$x \sin\omega = x' \sin(\omega - \theta) - y' \cos(\omega - \theta),$$

and
$$y \sin\omega = x' \sin\theta + y' \cos\theta.$$

These particular formulæ may easily be proved independently, by drawing the corresponding figures.

Ex. *Transform the equation* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *from rectangular axes to axes inclined at an angle* 2α, *the new axis of* x *being inclined at an angle* $-\alpha$ *to the old axes and* $\sin\alpha$ *being equal to* $\frac{b}{\sqrt{a^2+b^2}}$.

Here $\theta = -\alpha$ and $\omega' = 2\alpha$, so that the formulæ of transformation (1) become

$$x = (x' + y') \cos\alpha \text{ and } y = (y' - x') \sin\alpha.$$

Since $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, we have $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, and hence the given equation becomes

$$\frac{(x'+y')^2}{a^2+b^2} - \frac{(y'-x')^2}{a^2+b^2} = 1,$$

i.e.
$$x'y' = \tfrac{1}{4}(a^2+b^2).$$

***134.** *The degree of an equation is unchanged by any transformation of coordinates.*

For the most general form of transformation is found by combining together Arts. 128 and 132. Hence the most general formulæ of transformation are

$$x = h + x' \frac{\sin(\omega - \theta)}{\sin\omega} + y' \frac{\sin(\omega - \omega' - \theta)}{\sin\omega},$$

and
$$y = k + x' \frac{\sin\theta}{\sin\omega} + y' \frac{\sin(\omega' + \theta)}{\sin\omega}.$$

For x and y we have therefore to substitute expressions in x' and y' of the first degree, so that by this substitution the degree of the equation cannot be raised.

Neither can, by this substitution, the degree be lowered. For, if it could, then, by transforming back again, the degree would be raised and this we have just shewn to be impossible.

***135.** *If by any change of axes, without change of origin, the quantity $ax^2+2hxy+by^2$ become*

$$a'x'^2+2h'x'y'+b'y'^2,$$

the axes in each case being rectangular, to prove that

$$a+b=a'+b', \text{ and } ab-h^2=a'b'-h'^2.$$

By Art. 129, the new axis of x being inclined at an angle θ to the old axis, we have to substitute

$$x'\cos\theta-y'\sin\theta \text{ and } x'\sin\theta+y'\cos\theta$$

for x and y respectively.

Hence $ax^2+2hxy+by^2$

$$=a(x'\cos\theta-y'\sin\theta)^2+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+b(x'\sin\theta+y'\cos\theta)^2$$
$$=x'^2[a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]$$
$$+2x'y'[-a\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)+b\cos\theta\sin\theta]$$
$$+y'^2[a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta].$$

We then have

$$a'=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta$$
$$=\tfrac{1}{2}[(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta]\dots\dots\dots\dots(1),$$
$$b'=a\sin^2\theta-2h\cos\theta\sin\theta+b\cos^2\theta$$
$$=\tfrac{1}{2}[(a+b)-(a-b)\cos 2\theta-2h\sin 2\theta]\dots\dots\dots\dots(2),$$

and $$h'=-a\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta)+b\cos\theta\sin\theta$$
$$=\tfrac{1}{2}[2h\cos 2\theta-(a-b)\sin 2\theta]\dots\dots\dots\dots\dots\dots(3).$$

By adding (1) and (2), we have $a'+b'=a+b$.

Also, by multiplying them, we have

$$4a'b'=(a+b)^2-\{(a-b)\cos 2\theta+2h\sin 2\theta\}^2.$$

Hence $4a'b'-4h'^2$

$$=(a+b)^2-[\{2h\sin 2\theta+(a-b)\cos 2\theta\}^2+\{2h\cos 2\theta-(a-b)\sin 2\theta\}^2]$$
$$=(a+b)^2-[(a-b)^2+4h^2]=4ab-4h^2,$$

so that $$a'b'-h'^2=ab-h^2.$$

136. *To find the angle through which the axes must be turned so that the expression $ax^2+2hxy+by^2$ may become an expression in which there is no term involving $x'y'$.*

Assuming the work of the previous article the coefficient of $x'y'$ vanishes if h' be zero, or, from equation (3), if

$$2h\cos 2\theta = (a-b)\sin 2\theta,$$

i.e. if
$$\tan 2\theta = \frac{2h}{a-b}.$$

The required angle is therefore

$$\tfrac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right).$$

***137.** The proposition of Art. 135 is a particular case, when the axes are rectangular, of the following more general proposition.

If by any change of axes, without change of origin, the quantity $ax^2 + 2hxy + by^2$ becomes $a'x^2 + 2h'xy + b'y^2$, then

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'},$$

and
$$\frac{ab-h^2}{\sin^2\omega} = \frac{a'b'-h'^2}{\sin^2\omega'},$$

ω and ω' being the angles between the original and final pairs of axes.

Let the coordinates of any point P, referred to the original axes, be x and y and, referred to the final axes, let them be x' and y'.

By Art. 20 the square of the distance between P and the origin is $x^2 + 2xy\cos\omega + y^2$, referred to the original axes, and $x'^2 + 2x'y'\cos\omega' + y'^2$, referred to the final axes.

We therefore always have

$$x^2 + 2xy\cos\omega + y^2 = x'^2 + 2x'y'\cos\omega' + y'^2 \ldots\ldots(1).$$

Also, by supposition, we have

$$ax^2 + 2hxy + by^2 = a'x'^2 + 2h'x'y' + b'y'^2 \ldots\ldots(2).$$

Multiplying (1) by λ and adding it to (2), we therefore have

$$x^2(a+\lambda) + 2xy(h+\lambda\cos\omega) + y^2(b+\lambda)$$
$$= x'^2(a'+\lambda) + 2x'y'(h'+\lambda\cos\omega') + y'^2(b'+\lambda) \ldots(3).$$

If then any value of λ makes the left-hand side of (3) a perfect square, the same value must make the right-hand side also a perfect square.

But the values of λ which make the left-hand a perfect square are given by the condition

$$(h+\lambda\cos\omega)^2 = (a+\lambda)(b+\lambda),$$

i.e. by

$$\lambda^2(1-\cos^2\omega)+\lambda(a+b-2h\cos\omega)+ab-h^2=0,$$

i.e. by $$\lambda^2+\lambda\frac{a+b-2h\cos\omega}{\sin^2\omega}+\frac{ab-h^2}{\sin^2\omega}=0 \text{ .... } (4).$$

In a similar manner the values of λ which make the right-hand side of (3) a perfect square are given by the equation

$$\lambda^2+\lambda\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}+\frac{a'b'-h'^2}{\sin^2\omega'}=0 \text{ ...... } (5).$$

Since the values of λ given by equation (4) are the same as the values of λ given by (5), the two equations (4) and (5) must be the same.

Hence we have

$$\frac{\mathbf{a+b-2h\cos\omega}}{\mathbf{\sin^2\omega}}=\frac{\mathbf{a'+b'-2h'\cos\omega'}}{\mathbf{\sin^2\omega'}}$$

and $$\frac{\mathbf{ab-h^2}}{\mathbf{\sin^2\omega}}=\frac{\mathbf{a'b'-h'^2}}{\mathbf{\sin^2\omega'}}.$$

EXAMPLES. XVI.

1. The equation to a straight line referred to axes inclined at 30° to one another is $y=2x+1$. Find its equation referred to axes inclined at 45°, the origin and axis of x being unchanged.

2. Transform the equation $2x^2+3\sqrt{3}xy+3y^2=2$ from axes inclined at 30° to rectangular axes, the axis of x remaining unchanged.

3. Transform the equation $x^2+xy+y^2=8$ from axes inclined at 60° to axes bisecting the angles between the original axes.

4. Transform the equation $y^2+4y\cot\alpha-4x=0$ from rectangular axes to oblique axes meeting at an angle α, the axis of x being kept the same.

5. If x and y be the coordinates of a point referred to a system of oblique axes, and x' and y' be its coordinates referred to another system of oblique axes with the same origin, and if the formulæ of transformation be

$$x=mx'+ny' \text{ and } y=m'x'+n'y',$$

prove that $$\frac{m^2+m'^2-1}{n^2+n'^2-1}=\frac{mm'}{nn'}.$$

CHAPTER VIII.

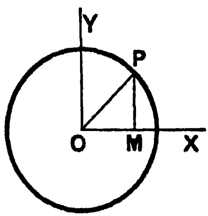
THE CIRCLE.

138. Def. A circle is the locus of a point which moves so that its distance from a fixed point, called the centre, is equal to a given distance. The given distance is called the radius of the circle.

139. *To find the equation to a circle, the axes of coordinates being two straight lines through its centre at right angles.*

Let O be the centre of the circle and let a be its radius.

Let OX and OY be the axes of coordinates.



Let P be any point on the circumference of the circle, and let its coordinates be x and y.

Draw PM perpendicular to OX and join OP.

Then (Euc. I. 47)

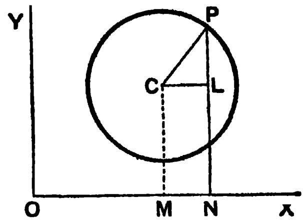
$$OM^2 + MP^2 = a^2,$$

i.e.
$$\mathbf{x^2 + y^2 = a^2}.$$

This being the relation which holds between the coordinates of any point on the circumference is, by Art. 42, the required equation.

140. *To find the equation to a circle referred to any rectangular axes.*

Let OX and OY be the two rectangular axes.



Let C be the centre of the circle and a its radius.

Take any point P on the circumference and draw perpendiculars CM and PN upon OX; let P be the point (x, y).

Draw CL perpendicular to NP.

Let the coordinates of C be h and k; these are supposed to be known.

We have $CL = MN = ON - OM = x - h$,

and $LP = NP - NL = NP - MC = y - k$.

Hence, since $CL^2 + LP^2 = CP^2$,

we have $$(\mathbf{x}-\mathbf{h})^2 + (\mathbf{y}-\mathbf{k})^2 = \mathbf{a}^2 \qquad \text{...........(1)}.$$

This is the required equation.

Ex. The equation to the circle, whose centre is the point $(-3, 4)$ and whose radius is 7, is

$$(x+3)^2 + (y-4)^2 = 7^2,$$

i.e. $$x^2 + y^2 + 6x - 8y = 24.$$

141. Some particular cases of the preceding article may be noticed:

(*a*) Let the origin O be on the circle so that, in this case,

$$OM^2 + MC^2 = a^2,$$

i.e. $$h^2 + k^2 = a^2.$$

The equation (1) then becomes

$$(x-h)^2 + (y-k)^2 = h^2 + k^2,$$

i.e. $$x^2 + y^2 - 2hx - 2ky = 0.$$

(β) Let the origin be not on the curve, but let the centre lie on the axis of x. In this case $k=0$, and the equation becomes

$$(x-h)^2 + y^2 = a^2.$$

(γ) Let the origin be on the curve and let the axis of x be a diameter. We now have $k=0$ and $a=h$, so that the equation becomes

$$x^2 + y^2 - 2hx = 0.$$

(δ) By taking O at C, and thus making both h and k zero, we have the case of Art. 139.

(c) The circle will touch the axis of x if MC be equal to the radius, *i.e.* if $k=a$.

The equation to a circle touching the axis of x is therefore

$$x^2+y^2-2hx-2ky+h^2=0.$$

Similarly, one touching the axis of y is

$$x^2+y^2-2hx-2ky+k^2=0.$$

142. *To prove that the equation*

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots\ldots(1),$$

always represents a circle for all values of g, f, and c, and to find its centre and radius. [The axes are assumed to be rectangular.]

This equation may be written

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c,$$

i.e.
$$(x+g)^2+(y+f)^2=\{\sqrt{g^2+f^2-c}\}^2.$$

Comparing this with the equation (1) of Art. 140, we see that the equations are the same if

$$h=-g,\quad k=-f,\ \text{and}\ a=\sqrt{g^2+f^2-c}.$$

Hence (1) represents a circle whose centre is the point $(-g, -f)$, and whose radius is $\sqrt{g^2+f^2-c}$.

If $g^2+f^2>c$, the radius of this circle is real.

If $g^2+f^2=c$, the radius vanishes, *i.e.* the circle becomes a point coinciding with the point $(-g, -f)$. Such a circle is called a point-circle.

If $g^2+f^2<c$, the radius of the circle is imaginary. In this case the equation does not represent any real geometrical locus. It is better not to say that the circle does not exist, but to say that it is a circle with a real centre and an imaginary radius.

Ex. 1. The equation $x^2+y^2+4x-6y=0$ can be written in the form

$$(x+2)^2+(y-3)^2=13=(\sqrt{13})^2,$$

and therefore represents a circle whose centre is the point $(-2, 3)$ and whose radius is $\sqrt{13}$.

Ex. 2. The equation $45x^2+45y^2-60x+36y+19=0$ is equivalent to

$$x^2+y^2-\tfrac{4}{3}x+\tfrac{4}{5}y=-\tfrac{19}{45},$$

i. e. $$(x-\tfrac{2}{3})^2+(y+\tfrac{2}{5})^2=\tfrac{4}{9}+\tfrac{4}{25}-\tfrac{19}{45}=\tfrac{41}{225},$$

and therefore represents a circle whose centre is the point $(\tfrac{2}{3}, -\tfrac{2}{5})$ and whose radius is $\frac{\sqrt{41}}{15}$.

143. *Condition that the general equation of the second degree may represent a circle.*

The equation (1) of the preceding article, multiplied by any arbitrary constant, is a particular case of the general equation of the second degree (Art. 114) in which there is no term containing xy and in which the coefficients of x^2 and y^2 are equal.

The general equation of the second degree in rectangular coordinates therefore represents a circle **if the coefficients of x^2 and y^2 be the same and if the coefficient of xy be zero.**

144. The equation (1) of Art. 142 is called the **general equation of a circle,** since it can, by a proper choice of g, f, and c, be made to represent *any* circle.

The three constants g, f, and c in the general equation correspond to the geometrical fact that a circle can be found to satisfy three independent geometrical conditions and no more. Thus a circle is determined when three points on it are given, or when it is required to touch three straight lines.

145. *To find the equation to the circle which is described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.*

Let A be the point (x_1, y_1) and B be the point (x_2, y_2), and let the coordinates of any point P on the circle be h and k.

The equation to AP is (Art. 62)

$$y-y_1=\frac{k-y_1}{h-x_1}(x-x_1) \qquad \qquad (1),$$

and the equation to BP is

$$y-y_2=\frac{k-y_2}{h-x_2}(x-x_2) \qquad \qquad (2).$$

But, since APB is a semicircle, the angle APB is a right angle, and hence the straight lines (1) and (2) are at right angles.

Hence, by Art. 69, we have

$$\frac{k-y_1}{h-x_1} \cdot \frac{k-y_2}{h-x_2} = -1,$$

i.e. $$(h-x_1)(h-x_2)+(k-y_1)(k-y_2)=0.$$

But this is the condition that the point (h, k) may lie on the curve whose equation is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

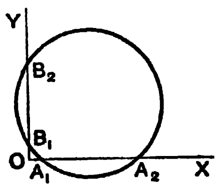
This therefore is the required equation.

146. *Intercepts made on the axes by the circle whose equation is*

$$ax^2+ay^2+2gx+2fy+c=0 \text{ ....(1).}$$

The abscissæ of the points where the circle (1) meets the axis of x, *i.e.* $y=0$, are given by the equation

$$ax^2+2gx+c=0 \text{ ....(2).}$$



The roots of this equation being x_1 and x_2, we have

$$x_1+x_2=-\frac{2g}{a},$$

and $$x_1x_2=\frac{c}{a}. \quad \text{(Art. 2.)}$$

Hence

$$A_1A_2=x_2-x_1=\sqrt{(x_1+x_2)^2-4x_1x_2}$$

$$=\sqrt{\frac{4g^2}{a^2}-\frac{4c}{a}}=2\frac{\sqrt{g^2-ac}}{a}.$$

Again, the roots of the equation (2) are both imaginary if $g^2<ac$. In this case the circle does not meet the axis of x in real points, *i.e.* geometrically it does not meet the axis of x at all.

The circle will touch the axis of x if the intercept A_1A_2 be just zero, *i.e.* if $g^2=ac$.

It will meet the axis of x in two points lying on opposite sides of the origin O if the two roots of the equation (2) are of opposite signs, *i.e.* if c be negative.

147. Ex. 1. *Find the equation to the circle which passes through the points* (1, 0), (0, −6), *and* (3, 4).

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0 \text{ .... (1).}$$

Since the three points, whose coordinates are given, satisfy this equation, we have

$$1+2g+c=0 \text{.................. .... (2),}$$

$$36-12f+c=0 \text{.............................. (3),}$$

and $$25+6g+8f+c=0 \text{.............................. (4).}$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g+12f=35,$$

and

$$6g+20f=11.$$

Hence $\qquad f=\frac{47}{8}$ and $g=-\frac{71}{4}$.

Equation (2) then gives $\qquad c=\frac{69}{2}$.

Substituting these values in (1) the required equation is

$$4x^2+4y^2-142x+47y+138=0.$$

Ex. 2. *Find the equation to the circle which touches the axis of y at a distance $+4$ from the origin and cuts off an intercept 6 from the axis of x.*

Any circle is $\qquad x^2+y^2+2gx+2fy+c=0.$

This meets the axis of y in points given by

$$y^2+2fy+c=0.$$

The roots of this equation must be equal and each equal to 4, so that it must be equivalent to $(y-4)^2=0$.

Hence $\qquad 2f=-8$, and $c=16$.

The equation to the circle is then

$$x^2+y^2+2gx-8y+16=0.$$

This meets the axis of x in points given by

$$x^2+2gx+16=0,$$

i.e. at points distant

$$-g+\sqrt{g^2-16} \text{ and } -g-\sqrt{g^2-16}.$$

Hence $\qquad 6=2\sqrt{g^2-16}.$

Therefore $g=\pm 5$, and the required equation is

$$x^2+y^2\pm 10x-8y+16=0.$$

There are therefore two circles satisfying the given conditions. This is geometrically obvious.

EXAMPLES. XVII.

Find the equation to the circle

1. Whose radius is 3 and whose centre is $(-1, 2)$.
2. Whose radius is 10 and whose centre is $(-5, -6)$.
3. Whose radius is $a+b$ and whose centre is $(a, -b)$.
4. Whose radius is $\sqrt{a^2-b^2}$ and whose centre is $(-a, -b)$.

Find the coordinates of the centres and the radii of the circles whose equations are

5. $x^2+y^2-4x-8y=41$ $\qquad$ 6. $3x^2+3y^2-5x-6y+4=0.$

7. $x^2+y^2=k(x+k)$. 8. $x^2+y^2=2gx-2fy$.

9. $\sqrt{1+m^2}(x^2+y^2)-2cx-2mcy=0$.

Draw the circles whose equations are

10. $x^2+y^2=2ay$. 11. $3x^2+3y^2=4x$.

12. $5x^2+5y^2=2x+3y$.

13. Find the equation to the circle which passes through the points $(1, -2)$ and $(4, -3)$ and which has its centre on the straight line $3x+4y=7$.

14. Find the equation to the circle passing through the points $(0, a)$ and (b, h), and having its centre on the axis of x.

Find the equations to the circles which pass through the points

15. $(0, 0)$, $(a, 0)$, and $(0, b)$. 16. $(1, 2)$, $(3, -4)$, and $(5, -6)$.

17. $(1, 1)$, $(2, -1)$, and $(3, 2)$. 18. $(5, 7)$, $(8, 1)$, and $(1, 3)$.

19. (a, b), $(a, -b)$, and $(a+b, a-b)$.

20. $ABCD$ is a square whose side is a 4; taking AB and AD as axes, prove that the equation to the circle circumscribing the square is

$$x^2+y^2=a(x+y).$$

4

21. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes.

22. Find the equation to the circle passing through the origin and the points (a, b) and (b, a). Find the lengths of the chords that it cuts off from the axes.

(2, 5) 5, 2

23. Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

2, 5

24. Find the equation to the circle, of radius a 3, which passes through the two points on the axis of x which are at a distance b from the origin.

Find the equation to the circle which

25. touches each axis at a distance 5 from the origin.

26. touches each axis and is of radius a. 6

27. touches both axes and passes through the point $(-2, -3)$.

28. touches the axis of x and passes through the two points $(1, -2)$ and $(3, -4)$.

29. touches the axis of y at the origin and passes through the point (h, k).

(1, -2)

30. touches the axis of x at a distance 3 from the origin and intercepts a distance 6 on the axis of y.

31. Points $(1, 0)$ and $(2, 0)$ are taken on the axis of x, the axes being rectangular. On the line joining these points an equilateral triangle is described, its vertex being in the positive quadrant. Find the equations to the circles described on its sides as diameters.

32. If $y = mx$ be the equation of a chord of a circle whose radius is a, the origin of coordinates being one extremity of the chord and the axis of x being a diameter of the circle, prove that the equation of a circle of which this chord is the diameter is

$$(1+m^2)(x^2+y^2) - 2a(x+my) = 0.$$

33. Find the equation to the circle passing through the points $(12, 43)$, $(18, 39)$, and $(42, 3)$ and prove that it also passes through the points $(-54, -69)$ and $(-81, -38)$.

34. Find the equation to the circle circumscribing the quadrilateral formed by the straight lines

$$2x+3y=2, \quad 3x-2y=4, \quad x+2y=3, \quad \text{and} \quad 2x-y=3.$$

35. Prove that the equation to the circle of which the points (x_1, y_1) and (x_2, y_2) are the ends of a chord of a segment containing an angle θ is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)$$
$$\pm \cot\theta\,[(x-x_1)(y-y_2)-(x-x_2)(y-y_1)]=0.$$

36. Find the equations to the circles in which the line joining the points (a, b) and $(b, -a)$ is a chord subtending an angle of $45°$ at any point on its circumference.

148. Tangent. In Geometry the tangent at any point of a circle is defined to be a straight line which meets the circle there, but, being produced, does not cut it; this tangent is shown to be always perpendicular to the radius drawn from the centre to the point of contact.

From this property may be deduced the equation to the tangent at any point (x', y') of the circle $x^2+y^2=a^2$.

For let the point P (Fig. Art. 139) be the point (x', y').

The equation to any straight line passing through P is, by Art. 62,

$$y-y'=m(x-x') \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also the equation to OP is

$$y=\frac{y'}{x'}x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The straight lines (1) and (2) are at right angles, *i.e.* the line (1) is a tangent, if

$$m \times \frac{y'}{x'} = -1, \qquad \text{(Art. 69)}$$

i.e. if

$$m = -\frac{x'}{y'}.$$

Substituting this value of m in (1), the equation of the tangent at (x', y') is

$$y - y' = -\frac{x'}{y'}(x - x'),$$

i.e.

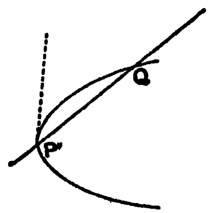
$$xx' + yy' = x'^2 + y'^2 \qquad \text{(3)}.$$

But, since (x', y') lies on the circle, we have $x'^2 + y'^2 = a^2$, and the required equation is then

$$\mathbf{xx' + yy' = a^2}.$$

149. In the case of most curves it is impossible to give a simple construction for the tangent as in the case of the circle. It is therefore necessary, in general, to give a different definition.

Tangent. Def. Let P and Q be any two points, near to one another, on any curve.



Join PQ; then PQ is called a secant.

The position of the line PQ when the point Q is taken indefinitely close to, and ultimately coincident with, the point P is called the tangent at P.

The student may better appreciate this definition, if he conceive the curve to be made up of a succession of very small points (much smaller than could be made by the finest conceivable drawing pen) packed close to one another along the curve. The tangent at P is then the straight line joining P and the next of these small points.

150. *To find the equation of the tangent at the point (x', y') of the circle $x^2 + y^2 = a^2$.*

Let P be the given point and Q a point (x'', y'') lying on the curve and close to P.

The equation to PQ is then

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \qquad \text{.............. (1).}$$

Since both (x', y') and (x'', y'') lie on the circle, we have

$$x'^2 + y'^2 = a^2,$$

and

$$x''^2 + y''^2 = a^2.$$

By subtraction, we have

$$x''^2 - x'^2 + y''^2 - y'^2 = 0,$$

i.e.

$$(x'' - x')(x'' + x') + (y'' - y')(y'' + y') = 0,$$

i.e.

$$\frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'}.$$

Substituting this value in (1), the equation to PQ is

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x') \qquad \text{..............(2).}$$

Now let Q be taken very close to P, so that it ultimately coincides with P, *i.e.* put $x'' = x'$ and $y'' = y'$.

Then (2) becomes

$$y - y' = -\frac{2x'}{2y'}(x - x'),$$

i.e.

$$yy' + xx' = x'^2 + y'^2 = a^2.$$

The required equation is therefore

$$\mathbf{xx' + yy' = a^2} \qquad \text{....................(3).}$$

It will be noted that the equation to the tangent found in this article coincides with the equation found from the geometrical definition in Art. 148.

Our definition of a tangent and the geometrical definition therefore give the same straight line in the case of a circle.

151. *To obtain the equation of the tangent at any point (x', y') lying on the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Let P be the given point and Q a point (x'', y'') lying on the curve close to P.

The equation to PQ is therefore

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \dots\dots\dots\dots\dots (1).$$

Since both (x', y') and (x'', y'') lie on the circle, we have

$$x'^2 + y'^2 + 2gx' + 2fy' + c = 0 \quad \dots\dots\dots\dots (2),$$

and

$$x''^2 + y''^2 + 2gx'' + 2fy'' + c = 0 \quad \dots\dots\dots (3).$$

By subtraction, we have

$$x''^2 - x'^2 + y''^2 - y'^2 + 2g(x'' - x') + 2f(y'' - y') = 0,$$

i.e. $$(x'' - x')(x'' + x' + 2g) + (y'' - y')(y'' + y' + 2f) = 0,$$

i.e. $$\frac{y'' - y'}{x'' - x'} = -\frac{x'' + x' + 2g}{y'' + y' + 2f}.$$

Substituting this value in (1), the equation to PQ becomes

$$y - y' = -\frac{x'' + x' + 2g}{y'' + y' + 2f}(x - x') \quad \dots\dots\dots\dots (4).$$

Now let Q be taken very close to P, so that it ultimately coincides with P, *i.e.* put $x'' = x'$ and $y'' = y'$.

The equation (4) then becomes

$$y - y' = -\frac{x' + g}{y' + f}(x - x'),$$

i.e. $$y(y' + f) + x(x' + g) = y'(y' + f) + x'(x' + g)$$
$$= x'^2 + y'^2 + gx' + fy'$$
$$= -gx' - fy' - c,$$

by (2).

This may be written

$$\mathbf{xx' + yy' + g(x + x') + f(y + y') + c = 0}$$

which is the required equation.

152. The equation to the tangent at (x', y') is therefore obtained from that of the circle itself by substituting xx' for x^2, yy' for y^2, $x + x'$ for $2x$, and $y + y'$ for $2y$.

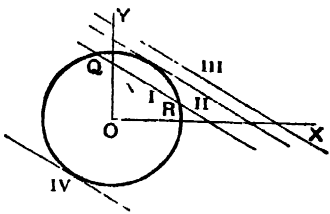
This is a particular case of a general rule which will be found to enable us to write down at sight the equation to the tangent at (x', y') to any of the curves with which we shall deal in this book.

153. *Points of intersection, in general, of the straight line*

$$y = mx + c \qquad\qquad (1),$$

with the circle

$$x^2 + y^2 = a^2 \qquad\qquad (2).$$



The coordinates of the points in which the straight line (1) meets (2) satisfy both equations (1) and (2).

If therefore we solve them as simultaneous equations we shall obtain the coordinates of the common point or points.

Substituting for y from (1) in (2), the abscissæ of the required points are given by the equation

$$x^2 + (mx + c)^2 = a^2,$$

i.e.
$$x^2 (1 + m^2) + 2mcx + c^2 - a^2 = 0 \qquad\qquad (3).$$

The roots of this equation are, by Art. 1, real, coincident, or imaginary, according as

$(2mc)^2 - 4 (1 + m^2) (c^2 - a^2)$ is positive, zero, or negative,

i.e. according as

$a^2 (1 + m^2) - c^2$ is positive, zero, or negative,

i.e. according as

c^2 is $<=$ or $> a^2 (1 + m^2)$.

In the figure the lines marked I, II, and III are all parallel, *i.e.* their equations all have the same "m."

The straight line I corresponds to a value of c^2 which is $< a^2(1+m^2)$ and it meets the circle in two real points.

The straight line III which corresponds to a value of c^2, $> a^2(1+m^2)$, does not meet the circle at all, or rather, as in Art. 108, this is better expressed by saying that it meets the circle in imaginary points.

The straight line II corresponds to a value of c^2, which is equal to $a^2(1+m^2)$, and meets the curve in two coincident points, *i.e.* is a tangent.

154. We can now obtain the length of the chord intercepted by the circle on the straight line (1). For, if x_1 and x_2 be the roots of the equation (3), we have

$$x_1 + x_2 = -\frac{2mc}{1+m^2}, \text{ and } x_1 x_2 = \frac{c^2 - a^2}{1+m^2}.$$

Hence

$$x_1 - x_2 = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = \frac{2}{1+m^2}\sqrt{m^2c^2 - (c^2-a^2)(1+m^2)}$$
$$= \frac{2}{1+m^2}\sqrt{a^2(1+m^2) - c^2}.$$

If y_1 and y_2 be the ordinates of Q and R we have, since these points are on (1),

$$y_1 - y_2 = (mx_1 + c) - (mx_2 + c) = m(x_1 - x_2).$$

Hence

$$QR = \sqrt{(y_1-y_2)^2 + (x_1-x_2)^2} = \sqrt{1+m^2}\,(x_1 - x_2)$$
$$= 2\sqrt{\frac{a^2(1+m^2) - c^2}{1+m^2}}.$$

In a similar manner we can consider the points of intersection of the straight line $y = mx + k$ with the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

155. *The straight line*

$$y = mx + a\sqrt{1+m^2}$$

is always a tangent to the circle

$$x^2 + y^2 = a^2.$$

As in Art. 153 the straight line

$$y = mx + c$$

meets the circle in two points which are coincident if

$$c = a\sqrt{1+m^2}.$$

But if a straight line meets the circle in two points which are indefinitely close to one another then, by Art. 149, it is a tangent to the circle.

The straight line $y = mx + c$ is therefore a tangent to the circle if

$$c = a\sqrt{1+m^2},$$

i.e. the equation to any tangent to the circle is

$$\mathbf{y = mx + a\sqrt{1+m^2}} \quad \text{.......... ...} \quad (1).$$

Since the radical on the right hand may have the + or − sign prefixed we see that corresponding to any value of m there are two tangents. They are marked II and IV in the figure of Art. 153.

156. The above result may also be deduced from the equation (3) of Art. 150, which may be written

$$y = -\frac{x'}{y'}x + \frac{a^2}{y'} \quad \text{....... ....} \quad (1).$$

Put $-\frac{x'}{y'} = m$, so that $x' = -my'$, and the relation $x'^2 + y'^2 = a^2$ gives

$$y'^2(m^2+1) = a^2, \ \ i.e. \ \ \frac{a}{y'} = \sqrt{1+m^2}.$$

The equation (1) then becomes

$$y = mx + a\sqrt{1+m^2}.$$

This is therefore the tangent at the point whose coordinates are

$$\frac{-ma}{\sqrt{1+m^2}} \text{ and } \frac{a}{\sqrt{1+m^2}}.$$

157. If we assume that a tangent to a circle is always perpendicular to the radius vector to the point of contact, the result of Art. 155 may be obtained in another manner.

For a tangent is a line whose perpendicular distance from the centre is equal to the radius.

The straight line $y = mx + c$ will therefore touch the circle if the perpendicular on it from the origin be equal to a, *i.e.* if

$$\frac{c}{\sqrt{1+m^2}} = a,$$

i.e. if $$c = a\sqrt{1+m^2}.$$

This method is not however applicable to any other curve besides the circle.

158. Ex. *Find the equations to the tangents to the circle*

$$x^2 + y^2 - 6x + 4y = 12$$

which are parallel to the straight line

$$4x + 3y + 5 = 0.$$

Any straight line parallel to the given one is

$$4x + 3y + C = 0 \qquad \text{.......................... (1).}$$

The equation to the circle is

$$(x-3)^2 + (y+2)^2 = 5^2.$$

The straight line (1), if it be a tangent, must be therefore such that its distance from the point $(3, -2)$ is equal to ± 5.

Hence $$\frac{12 - 6 + C}{\sqrt{4^2 + 3^2}} = \pm 5, \qquad \text{(Art. 75),}$$

so that $$C = -6 \pm 25 = 19 \text{ or } -31.$$

The required tangents are therefore

$$4x + 3y + 19 = 0 \text{ and } 4x + 3y - 31 = 0.$$

159. Normal. Def. The normal at any point P of a curve is the straight line which passes through P and is perpendicular to the tangent at P.

To find the equation to the normal at the point (x', y') of (1) *the circle*

$$x^2 + y^2 = a^2,$$

and (2) *the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

(1) The tangent at (x', y') is

$$xx' + yy' = a^2,$$

i.e. $$y = -\frac{x'}{y'}x + \frac{a^2}{y'}.$$

The equation to the straight line passing through (x', y') perpendicular to this tangent is

$$y - y' = m(x - x'),$$

where $$m \times \left(-\frac{x'}{y'}\right) = -1, \qquad \text{(Art. 69)},$$

i.e. $$m = \frac{y'}{x'}.$$

The required equation is therefore

$$y - y' = \frac{y'}{x'}(x - x'),$$

i.e. $$x'y - xy' = 0.$$

This straight line passes through the centre of the circle which is the point (0, 0).

If we assume the ordinary geometrical propositions the equation is at once written down, since the normal is the straight line joining (0, 0) to (x', y').

(2) The equation to the tangent at (x', y') to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$y = -\frac{x' + g}{y' + f}x - \frac{gx' + fy' + c}{y' + f}. \qquad \text{(Art. 151.)}$$

The equation to the straight line, passing through the point (x', y') and perpendicular to this tangent, is

$$y - y' = m(x - x'),$$

where $$m \times \left(-\frac{x' + g}{y' + f}\right) = -1, \qquad \text{(Art. 69)},$$

i.e. $$m = \frac{y' + f}{x' + g}.$$

The equation to the normal is therefore

$$y - y' = \frac{y' + f}{x' + g}(x - x'),$$

i.e. $$y(x' + g) - x(y' + f) + fx' - gy' = 0.$$

EXAMPLES. XVIII.

Write down the equation of the tangent to the circle

1. $x^2+y^2-3x+10y=15$ at the point $(4, -11)$.

2. $4x^2+4y^2-16x+24y=117$ at the point $(-4, -\frac{11}{2})$.

Find the equations to the tangents to the circle

3. $x^2+y^2=4$ which are parallel to the line $x+2y+3=0$.

4. $x^2+y^2+2gx+2fy+c=0$ which are parallel to the line
$$x+2y-6=0.$$

5. Prove that the straight line $y=x+c\sqrt{2}$ touches the circle $x^2+y^2=c^2$ and find its point of contact.

6. Find the condition that the straight line $cx-by+b^2=0$ may touch the circle $x^2+y^2=ax+by$ and find the point of contact.

7. Find whether the straight line $x+y=2+\sqrt{2}$ touches the circle
$$x^2+y^2-2x-2y+1=0.$$

8. Find the condition that the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10x$.

9. Find the value of p so that the straight line
$$x\cos\alpha+y\sin\alpha-p=0$$
may touch the circle
$$x^2+y^2-2ax\cos\alpha-2by\sin\alpha-a^2\sin^2\alpha=0.$$

10. Find the condition that the straight line $Ax+By+C=0$ may touch the circle
$$(x-a)^2+(y-b)^2=c^2.$$

11. Find the equation to the tangent to the circle $x^2+y^2=a^2$ which

(i) is parallel to the straight line $y=mx+c$,

(ii) is perpendicular to the straight line $y=mx+c$,

(iii) passes through the point $(b, 0)$,

and (iv) makes with the axes a triangle whose area is a^2.

12. Find the length of the chord joining the points in which the straight line
$$\frac{x}{a}+\frac{y}{b}=1$$
meets the circle $$x^2+y^2=r^2.$$

13. Find the equation to the circles which pass through the origin and cut off equal chords a from the straight lines $y=x$ and $y=-x$.

14. Find the equation to the straight lines joining the origin to the points in which the straight line $y = mx + c$ cuts the circle

$$x^2 + y^2 = 2ax + 2by.$$

Hence find the condition that these points may subtend a right angle at the origin.

Find also the condition that the straight line may touch the circle.

Find the equation to the circle which

15. has its centre at the point $(3, 4)$ and touches the straight line

$$5x + 12y = 1.$$

16. touches the axes of coordinates and also the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the centre being in the positive quadrant.

17. has its centre at the point $(1, -3)$ and touches the straight line $2x - y - 4 = 0$.

18. Find the general equation of a circle referred to two perpendicular tangents as axes.

19. Find the equation to a circle of radius r which touches the axis of y at a point distant h from the origin, the centre of the circle being in the positive quadrant.

Prove also that the equation to the other tangent which passes through the origin is

$$(r^2 - h^2)x + 2rhy = 0.$$

20. Find the equation to the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is

$$\alpha x + \beta y = 0.$$

21. Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ to touch the axis of y. Prove that they intersect at an angle $\tan^{-1}\frac{40}{9}$.

22. A circle passes through the points $(-1, 1)$, $(0, 6)$, and $(5, 5)$. Find the points on this circle the tangents at which are parallel to the straight line joining the origin to its centre.

160. *To shew that from any point there can be drawn two tangents, real or imaginary, to a circle.*

Let the equation to the circle be $x^2 + y^2 = a^2$, and let the given point be (x_1, y_1). [Fig. Art. 161.]

The equation to any tangent is, by Art. 155,

$$y = mx + a\sqrt{1 + m^2}.$$

If this pass through the given point (x_1, y_1) we have

$$y_1 = mx_1 + a\sqrt{1+m^2} \quad \text{................(1).}$$

This is the equation which gives the values of m corresponding to the tangents which pass through (x_1, y_1).

Now (1) gives

$$y_1 - mx_1 = a\sqrt{1+m^2},$$

i.e. $$y_1^2 - 2mx_1y_1 + m^2x_1^2 - a^2 + a^2m^2,$$

i.e. $$m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0 \quad \text{.... (2).}$$

The equation (2) is a quadratic equation and gives therefore two values of m (real, coincident, or imaginary) corresponding to any given values of x_1 and y_1. For each of these values of m we have a corresponding tangent.

The roots of (2) are, by Art. 1, real, coincident or imaginary according as

$(2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$ is positive, zero, or negative,

i.e. according as

$$a^2(-a^2 + x_1^2 + y_1^2) \text{ is positive, zero, or negative,}$$

i.e. according as $\qquad x_1^2 + y_1^2 \gtreqless a^2.$

If $x_1^2 + y_1^2 > a^2$, the distance of the point (x_1, y_1) from the centre is greater than the radius and hence it lies outside the circle.

If $x_1^2 + y_1^2 = a^2$, the point (x_1, y_1) lies on the circle and the two coincident tangents become the tangent at (x_1, y_1).

If $x_1^2 + y_1^2 < a^2$, the point (x_1, y_1) lies within the circle, and no tangents can then be geometrically drawn to the circle. It is however better to say that the tangents are imaginary

161. Chord of Contact. Def. If from any point T without a circle two tangents TP and TQ be drawn to the circle, the straight line PQ joining the points of contact is called the chord of contact of tangents from T.

To find the equation of the chord of contact of tangents drawn to the circle $x^2 + y^2 = a^2$ from the external point (x_1, y_1).

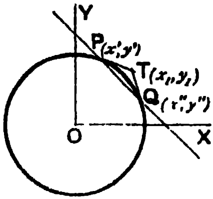
Let T be the point (x_1, y_1), and P and Q the points (x', y') and (x'', y'') respectively.

The tangent at P is

$$xx' + yy' = a^2 \quad \ldots\ldots (1),$$

and that at Q is

$$xx'' + yy'' = a^2 \quad \ldots\ldots (2).$$



Since these tangents pass through T, its coordinates (x_1, y_1) must satisfy both (1) and (2).

Hence $$x_1x' + y_1y' = a^2 \quad \ldots\ldots (3),$$

and $$x_1x'' + y_1y'' = a^2 \quad \ldots\ldots (4).$$

The equation to PQ is then

$$\mathbf{xx_1 + yy_1 = a^2} \quad \ldots\ldots (5).$$

For, since (3) is true, it follows that the point (x', y'), *i.e.* P, lies on (5).

Also, since (4) is true, it follows that the point (x'', y''), *i.e.* Q, lies on (5).

Hence both P and Q lie on the straight line (5), *i.e.* (5) is the equation to the required chord of contact.

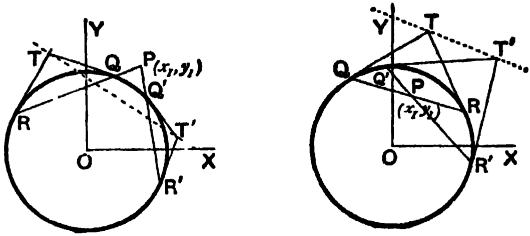
If the point (x_1, y_1) lie within the circle the argument of the preceding article will shew that the line joining the (imaginary) points of contact of the two (imaginary) tangents drawn from (x_1, y_1) is $xx_1 + yy_1 = a^2$.

We thus see, since this line is always real, that we may have a real straight line joining the imaginary points of contact of two imaginary tangents.

162. Pole and Polar. Def. If through a point P (within or without a circle) there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q and R is called the polar of P; also P is called the pole of the polar.

In the next article the locus will be proved to be a straight line.

163. *To find the equation to the polar of the point* (x_1, y_1) *with respect to the circle* $x^2 + y^2 = a^2$.



Let QR be any chord drawn through P and let the tangents at Q and R meet in the point T whose coordinates are (h, k).

Hence QR is the chord of contact of tangents drawn from the point (h, k) and therefore, by Art. 161, its equation is $xh + yk = a^2$.

Since this line passes through the point (x_1, y_1) we have

$$x_1h + y_1k = a^2 \dots\dots\dots\dots\dots\dots\dots(1).$$

Since the relation (1) is true it follows that the variable point (h, k) always lies on the straight line whose equation is

$$\mathbf{xx_1 + yy_1 = a^2} \dots\dots\dots\dots\dots\dots\dots(2).$$

Hence (2) is the polar of the point (x_1, y_1).

In a similar manner it may be proved that the polar of (x_1, y_1) with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

164. The equation (2) of the preceding article is the same as equation (5) of Art. 161. If, therefore, the point (x_1, y_1) be without the circle, as in the right-hand figure, the polar is the same as the chord of contact of the real tangents drawn through (x_1, y_1).

If the point (x_1, y_1) be on the circle, the polar coincides with the tangent at it. (Art. 150.)

If the point (x_1, y_1) be within the circle, then, as in Art. 161, the equation (2) is the line joining the (imaginary) points of contact of the two (imaginary) tangents that can be drawn from (x_1, y_1).

We see therefore that the polar might have been defined as follows:

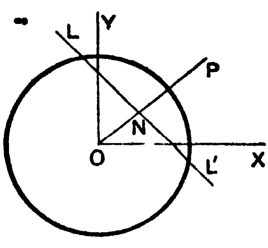
The polar of a given point is the straight line which passes through the (real or imaginary) points of contact of tangents drawn from the given point; also the pole of any straight line is the point of intersection of tangents at the points (real or imaginary) in which this straight line meets the circle.

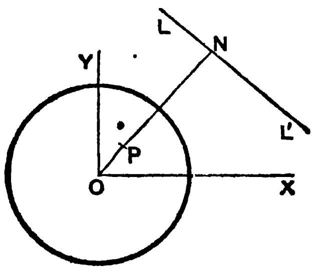
165. *Geometrical construction for the polar of a point.*

The equation to OP, which is the line joining (0, 0) to (x_1, y_1), is

$$y = \frac{y_1}{x_1} x,$$

$$xy_1 - x_1 y = 0 \quad \text{.....................(1)}.$$





Also the polar of P is

$$xx_1 + yy_1 = a^2 \quad \text{.....................(2)}.$$

By Art. 69, the lines (1) and (2) are perpendicular to one another. Hence OP is perpendicular to the polar of P.

Also the length $\quad OP = \sqrt{x_1^2 + y_1^2},$

and the perpendicular, ON, from O upon (2)

$$= \frac{a^2}{\sqrt{x_1^2 + y_1^2}}.$$

Hence the product $ON \,.\, OP = a^2$.

The polar of any point P is therefore constructed thus: Join OP and on it (produced if necessary) take a point N such that the rectangle $ON \,.\, OP$ is equal to the square of the radius of the circle.

Through N draw the straight line LL' perpendicular to OP; this is the polar required.

[It will be noted that the middle point N of any chord LL' lies on the line joining the centre to the pole of the chord.]

166. *To find the pole of a given line with respect to any circle.*

Let the equation to the given line be

$$Ax + By + C = 0 \quad\text{..................}(1).$$

(1) Let the equation to the circle be

$$x^2 + y^2 = a^2,$$

and let the required pole be (x_1, y_1).

Then (1) must be the equation to the polar of (x_1, y_1), *i.e.* it is the same as the equation

$$xx_1 + yy_1 - a^2 = 0 \quad\text{.................}(2).$$

Comparing equations (1) and (2), we have

$$\frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C},$$

so that $\qquad x_1 = -\frac{A}{C}a^2$ and $y_1 = -\frac{B}{C}a^2$.

The required pole is therefore the point

$$\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right).$$

(2) Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If (x_1, y_1) be the required pole, then (1) must be equivalent to the equation

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0, \qquad \text{(Art. 163)},$$

i.e. $$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0 \;\ldots\ldots (3).$$

Comparing (1) with (3), we therefore have

$$\frac{x_1 + g}{A} = \frac{y_1 + f}{B} = \frac{gx_1 + fy_1 + c}{C}.$$

By solving these equations we have the values of x_1 and y_1.

Ex. *Find the pole of the straight line*

$$9x + y - 28 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with respect to the circle

$$2x^2 + 2y^2 - 3x + 5y - 7 = 0. \;\ldots\ldots\ldots\ldots (2).$$

If (x_1, y_1) be the required point the line (1) must coincide with the polar of (x_1, y_1), whose equation is

$$2xx_1 + 2yy_1 - \tfrac{3}{2}(x + x_1) + \tfrac{5}{2}(y + y_1) - 7 = 0,$$

i.e. $$x(4x_1 - 3) + y(4y_1 + 5) - 3x_1 + 5y_1 - 14 = 0 \;\ldots\ldots\ldots (3).$$

Since (1) and (3) are the same, we have

$$\frac{4x_1 - 3}{9} = \frac{4y_1 + 5}{1} = \frac{-3x_1 + 5y_1 - 14}{-28}.$$

Hence $$x_1 = 9y_1 + 12,$$

and $$3x_1 - 117y_1 = 126.$$

Solving these equations we have $x_1 = 3$ and $y_1 = -1$, so that the required point is $(3, -1)$.

167. *If the polar of a point P pass through a point T, then the polar of T passes through P.*

Let P and T be the points (x_1, y_1) and (x_2, y_2) respectively. (Fig. Art. 163.)

The polar of (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

This straight line passes through the point T if

$$x_2x_1 + y_2y_1 = a^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the relation (1) is true it follows that the point (x_1, y_1), *i.e.* P, lies on the straight line $xx_2 + yy_2 = a^2$, which is the polar of (x_2, y_2), *i.e.* T, with respect to the circle.

Hence the proposition.

Cor. The intersection, T, of the polars of two points, P and Q, is the pole of the line PQ.

168. *To find the length of the tangent that can be drawn from the point* (x_1, y_1) *to the circles*

$$(1) \quad x^2 + y^2 = a^2,$$

and

$$(2) \quad x^2 + y^2 + 2gx + 2fy + c = 0.$$

If T be an external point (Fig. Art. 163), TQ a tangent and O the centre of the circle, then TQO is a right angle and hence

$$TQ^2 = OT^2 - OQ^2.$$

(1) If the equation to the circle be $x^2 + y^2 = a^2$, O is the origin, $OT^2 = x_1^2 + y_1^2$, and $OQ^2 = a^2$.

Hence $$TQ^2 = x_1^2 + y_1^2 - a^2.$$

(2) Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

i.e. $$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

In this case O is the point $(-g, -f)$ and

$$OQ^2 = (\text{radius})^2 = g^2 + f^2 - c.$$

Hence $$OT^2 = [x_1 - (-g)]^2 + [y_1 - (-f)]^2 \quad \text{(Art. 20)}.$$
$$= (x_1 + g)^2 + (y_1 + f)^2.$$

Therefore $$TQ^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

In each case we see that (the equation to the circle being written so that the coefficients of x^2 and y^2 are each unity, and the right-hand member zero), the square of the length of the tangent drawn to the circle from the point (x_1, y_1) is obtained by substituting x_1 and y_1 for x and y in the left-hand member of the equation.

***169.** *To find the equation to the pair of tangents that can be drawn from the point* (x_1, y_1) *to the circle* $x^2 + y^2 = a^2$.

Let (h, k) be any point on either of the tangents from (x_1, y_1).

Since any straight line touches a circle if the perpendicular on it from the centre is equal to the radius, the perpendicular from the origin upon the line joining (x_1, y_1) to (h, k) must be equal to a.

The equation to the straight line joining these two points is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1),$$

i.e. $$y(h - x_1) - x(k - y_1) + kx_1 - hy_1 = 0.$$

Hence $$\frac{kx_1 - hy_1}{\sqrt{(h - x_1)^2 + (k - y_1)^2}} = a,$$

so that $$(kx_1 - hy_1)^2 = a^2[(h - x_1)^2 + (k - y_1)^2].$$

Therefore the point (h, k) always lies on the locus

$$(x_1 y - x y_1)^2 = a^2[(x - x_1)^2 + (y - y_1)^2] \quad \text{........(1)}.$$

This therefore is the required equation.

The equation (1) may be written in the form

$$x^2(y_1^2 - a^2) + y^2(x_1^2 - a^2) - a^2(x_1^2 + y_1^2)$$
$$= 2xyx_1y_1 - 2a^2xx_1 - 2a^2yy_1,$$

i.e. $$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = x^2x_1^2 + y^2y_1^2 + a^4 + 2xyx_1y_1$$
$$- 2a^2xx_1 - 2a^2yy_1 = (xx_1 + yy_1 - a^2)^2 \quad \text{..........(2)}.$$

***170.** In a later chapter we shall obtain the equation to the pair of tangents to any curve of the second degree in a form analogous to that of equation (2) of the previous article.

Similarly the equation to the pair of tangents that can be drawn from (x_1, y_1) to the circle

$$(x - f)^2 + (y - g)^2 = a^2$$

is $$\{(x-f)^2 + (y-g)^2 - a^2\}\{(x_1 - f)^2 + (y_1 - g)^2 - a^2\}$$
$$= \{(x-f)(x_1 - f) + (y-g)(y_1 - g) - a^2\}^2 \quad \text{......(1)}.$$

If the equation to the circle be given in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

the equation to the tangents is, similarly,

$$(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$$
$$= [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2 \quad \text{......(2)}.$$

EXAMPLES. XIX.

Find the polar of the point

1. (1, 2) with respect to the circle $x^2+y^2=7$.

2. (4, −1) with respect to the circle $2x^2+2y^2=11$.

3. (−2, 3) with respect to the circle

$$x^2+y^2-4x-6y+5=0.$$

4. (5, −½) with respect to the circle

$$3x^2+3y^2-7x+8y-9=0.$$

5. $(a, -b)$ with respect to the circle

$$x^2+y^2+2ax-2by+a^2-b^2=0.$$

Find the pole of the straight line

6. $x+2y=1$ with respect to the circle $x^2+y^2=5$.

7. $2x-y=6$ with respect to the circle $5x^2+5y^2=9$.

8. $2x+y+12=0$ with respect to the circle

$$x^2+y^2-4x+3y-1=0.$$

9. $48x-54y+53=0$ with respect to the circle

$$3x^2+3y^2+5x-7y+2=0.$$

10. $ax+by+3a^2+3b^2=0$ with respect to the circle

$$x^2+y^2+2ax+2by=a^2+b^2.$$

11. Tangents are drawn to the circle $x^2+y^2=12$ at the points where it is met by the circle $x^2+y^2-5x+3y-2=0$; find the point of intersection of these tangents.

12. Find the equation to that chord of the circle $x^2+y^2=81$ which is bisected at the point (−2, 3), and its pole with respect to the circle.

13. Prove that the polars of the point (1, −2) with respect to the circles whose equations are

$$x^2+y^2+6y+5=0 \text{ and } x^2+y^2+2x+8y+5=0$$

coincide; prove also that there is another point the polars of which with respect to these circles are the same and find its coordinates.

14. Find the condition that the chord of contact of tangents from the point (x', y') to the circle $x^2+y^2=a^2$ should subtend a right angle at the centre.

15. Prove that the distances of two points, P and Q, each from the polar of the other with respect to a circle, are to one another as the distances of the points from the centre of the circle.

16. Prove that the polar of a given point with respect to any one of the circles $x^2+y^2-2kx+c^2=0$, where k is variable, always passes through a fixed point, whatever be the value of k.

17. Tangents are drawn from the point (h, k) to the circle $x^2+y^2=a^2$; prove that the area of the triangle formed by them and the straight line joining their points of contact is

$$\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}.$$

Find the lengths of the tangents drawn

18. to the circle $2x^2+2y^2=3$ from the point $(-2, 3)$.

19. to the circle $3x^2+3y^2-7x-6y=12$ from the point $(6, -7)$.

20. to the circle $x^2+y^2+2bx-3b^2=0$ from the point

$$(a+b,\ a-b).$$

21. Given the three circles

$$x^2+y^2-16x+60=0,$$
$$3x^2+3y^2-36x+81=0,$$

and
$$x^2+y^2-16x-12y+84=0,$$

find (1) the point from which the tangents to them are equal in length, and (2) this length.

22. The distances from the origin of the centres of three circles $x^2+y^2-2\lambda x=c^2$ (where c is a constant and λ a variable) are in geometrical progression; prove that the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are also in geometrical progression.

23. Find the equation to the pair of tangents drawn

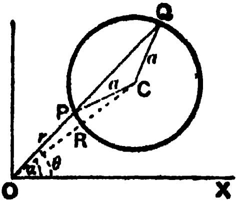
(1) from the point $(11, 3)$ to the circle $x^2+y^2=65$,

(2) from the point $(4, 5)$ to the circle

$$2x^2+2y^2-8x+12y+21=0.$$

171. *To find the general equation of a circle referred to polar coordinates.*

Let O be the origin, or pole, OX the initial line, C the centre and a the radius of the circle.



Let the polar coordinates of C be R and α, so that $OC=R$ and $\angle XOC=\alpha$.

Let a radius vector through O at an angle θ with the initial line cut the circle in P and Q. Let OP, or OQ, be r.

Then (*Trig.* Art. 164) we have

$$CP^2 = OC^2 + OP^2 - 2OC \,.\, OP \cos COP,$$

i.e. $$a^2 = R^2 + r^2 - 2Rr\cos(\theta - \alpha),$$

i.e. $$r^2 - 2Rr\cos(\theta - \alpha) + R^2 - a^2 = 0 \ldots\ldots\ldots\ldots(1).$$

This is the required polar equation.

172. *Particular cases of the general equation in polar coordinates.*

(1) Let the initial line be taken to go through the centre C. Then $\alpha = 0$, and the equation becomes

$$r^2 - 2Rr\cos\theta + R^2 - a^2 = 0.$$

(2) Let the pole O be taken on the circle, so that

$$R = OC = a.$$

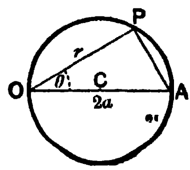
The general equation then becomes

$$r^2 - 2ar\cos(\theta - \alpha) = 0,$$

i.e. $$r = 2a\cos(\theta - \alpha).$$

(3) Let the pole be on the circle and also let the initial line pass through the centre of the circle. In this case

$$\alpha = 0, \quad \text{and} \quad R = a.$$



The general equation reduces then to the simple form $r = 2a\cos\theta$.

This is at once evident from the figure.

For, if OCA be a diameter, we have

$$OP = OA\cos\theta,$$

i.e. $$r = 2a\cos\theta.$$

173. The equation (1) of Art. 171 is a quadratic equation which, for any given value of θ, gives two values of r. These two values in the figure are OP and OQ.

If these two values be called r_1 and r_2, we have, from equation (1),

$$r_1 r_2 = \text{product of the roots} = R^2 - a^2,$$

i.e. $$OP \,.\, OQ = R^2 - a^2.$$

The value of the rectangle $OP \,.\, OQ$ is therefore the same for all values of θ. It follows that if we drew any other line through O to cut the circle in P_1 and Q_1 we should have $OP \,.\, OQ = OP_1 \,.\, OQ_1$.

This is the well-known geometrical proposition.

174. *Find the equation to the chord joining the points on the circle $r=2a\cos\theta$ whose vectorial angles are θ_1 and θ_2, and deduce the equation to the tangent at the point θ_1.*

The equation to any straight line in polar coordinates is (Art. 88)

$$p=r\cos(\theta-\alpha) \quad \ldots\ldots\ldots\ldots (1).$$

If this pass through the points $(2a\cos\theta_1, \theta_1)$ and $(2a\cos\theta_2, \theta_2)$, we have

$$2a\cos\theta_1\cos(\theta_1-\alpha)=p=2a\cos\theta_2\cos(\theta_2-\alpha) \ldots\ldots (2).$$

Hence $\quad \cos(2\theta_1-\alpha)+\cos\alpha=\cos(2\theta_2-\alpha)+\cos\alpha,$

i.e. $\quad 2\theta_1-\alpha=-(2\theta_2-\alpha),$

since θ_1 and θ_2 are not, in general, equal.

Hence $\quad \alpha=\theta_1+\theta_2,$

and then, from (2), $p=2a\cos\theta_1\cos\theta_2$.

On substitution in (1), the equation to the required chord is

$$r\cos(\theta-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2 \quad \ldots\ldots\ldots (3).$$

The equation to the tangent at the point θ_1 is found, as in Art. 150, by putting $\theta_2=\theta_1$ in equation (3).

We thus obtain as the equation to the tangent

$$r\cos(\theta-2\theta_1)=2a\cos^2\theta_1.$$

As in the foregoing article it could be shewn that the equation to the chord joining the points θ_1 and θ_2 on the circle $r=2a\cos(\theta-\gamma)$ is

$$r\cos[\theta-\theta_1-\theta_2+\gamma]=2a\cos(\theta_1-\gamma)\cos(\theta_2-\gamma)$$

and hence that the equation to the tangent at the point θ_1 is

$$r\cos(\theta-2\theta_1+\gamma)=2a\cos^2(\theta_1-\gamma).$$

EXAMPLES. XX.

1. Find the coordinates of the centre of the circle

$$r=A\cos\theta+B\sin\theta.$$

2. Find the polar equation of a circle, the initial line being a tangent. What does it become if the origin be on the circumference?

3. Draw the loci

(1) $r=a$; (2) $r=a\sin\theta$; (3) $r=a\cos\theta$; (4) $r=a\sec\theta$;
(5) $r=a\cos(\theta-\alpha)$; (6) $r=a\sec(\theta-\alpha)$.

4. Prove that the equations $r=a\cos(\theta-\alpha)$ and $r=b\sin(\theta-\alpha)$ represent two circles which cut at right angles.

5. Prove that the equation $r^2\cos\theta-ar\cos 2\theta-2a^2\cos\theta=0$ represents a straight line and a circle.

6. Find the polar equation to the circle described on the straight line joining the points (a, α) and (b, β) as diameter.

7. Prove that the equation to the circle described on the straight line joining the points $(1, 60^\circ)$ and $(2, 30^\circ)$ as diameter is

$$r^2 - r[\cos(\theta - 60^\circ) + 2\cos(\theta - 30^\circ)] + \sqrt{3} = 0.$$

8. Find the condition that the straight line

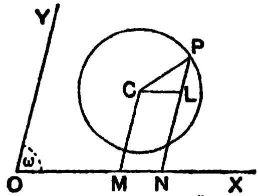
$$\frac{1}{r} = a\cos\theta + b\sin\theta$$

may touch the circle $\qquad r = 2c\cos\theta.$

175. *To find the general equation to a circle referred to oblique axes which meet at an angle* ω.

Let C be the centre and a the radius of the circle. Let the coordinates of C be (h, k) so that if CM, drawn parallel to the axis of y, meets OX in M, then

$$OM = h \text{ and } MC = k.$$



Let P be any point on the circle whose coordinates are x and y. Draw PN, the ordinate of P, and CL parallel to OX to meet PN in L.

Then $\qquad CL = MN = ON - OM = x - h,$

and $\qquad LP = NP - NL = NP - MC = y - k.$

Also $\quad \angle CLP = \angle ONP = 180^\circ - \angle PNX = 180^\circ - \omega.$

Hence, since $\quad CL^2 + LP^2 - 2CL \cdot LP\cos CLP = a^2,$

we have $\mathbf{(x - h)^2 + (y - k)^2 + 2(x - h)(y - k)\cos\omega = a^2}$,

i.e. $\quad x^2 + y^2 + 2xy\cos\omega - 2x(h + k\cos\omega) - 2y(k + h\cos\omega)$
$$+ h^2 + k^2 + 2hk\cos\omega = a^2.$$

The required equation is therefore found.

176. As in Art. 142 it may be shewn that the equation

$$x^2 + 2xy\cos\omega + y^2 + 2gx + 2fy + c = 0$$

represents a circle and its radius and centre found.

Ex. *If the axes be inclined at 60°, prove that the equation*

$$x^2+xy+y^2-4x-5y-2=0 \quad\text{.....................}(1)$$

represents a circle and find its centre and radius.

If ω be equal to 60°, so that $\cos\omega=\frac{1}{2}$, the equation of Art. 175 becomes

$$x^2+xy+y^2-x(2h+k)-y(2k+h)+h^2+k^2+hk=a^2.$$

This equation agrees with (1) if

$$2h+k=4 \quad\text{..............................}(2),$$

$$2k+h=5 \quad\text{..............................}(3),$$

and $$h^2+k^2+hk-a^2=-2 \quad\text{.....................}(4).$$

Solving (2) and (3), we have $h=1$ and $k=2$. Equation (4) then gives

$$a^2=h^2+k^2+hk+2=9,$$

so that $$a=3.$$

The equation (1) therefore represents a circle whose centre is the point (1, 2) and whose radius is 3, the axes being inclined at 60°.

EXAMPLES. XXI.

Find the inclinations of the axes so that the following equations may represent circles, and in each case find the radius and centre;

1. $x^2-xy+y^2-2gx-2fy=0.$

2. $x^2+\sqrt{3}xy+y^2-4x-6y+5=0.$

3. The axes being inclined at an angle ω, find the centre and radius of the circle

$$x^2+2xy\cos\omega+y^2-2gx-2fy=0.$$

4. The axes being inclined at 45°, find the equation to the circle whose centre is the point (2, 3) and whose radius is 4.

5. The axes being inclined at 60°, find the equation to the circle whose centre is the point $(-3, -5)$ and whose radius is 6.

6. Prove that the equation to a circle whose radius is a and which touches the axes of coordinates, which are inclined at an angle ω, is

$$x^2+2xy\cos\omega+y^2-2a(x+y)\cot\frac{\omega}{2}+a^2\cot^2\frac{\omega}{2}=0.$$

7. Prove that the straight line $y=mx$ will touch the circle

$$x^2+2xy\cos\omega+y^2+2gx+2fy+c=0$$

if $$(g+fm)^2=c(1+2m\cos\omega+m^2).$$

8. The axes being inclined at an angle ω, find the equation to the circle whose diameter is the straight line joining the points

$$(x', y') \text{ and } (x'', y'').$$

Coordinates of a point on a circle expressed in terms of one single variable.

177. If, in the figure of Art. 139, we put the angle MOP equal to α, the coordinates of the point P are easily seen to be $a\cos\alpha$ and $a\sin\alpha$.

These equations clearly satisfy equation (1) of that article.

The position of the point P is therefore known when the value of α is given, and it may be, for brevity, called "the point α."

With the ordinary Cartesian coordinates we have to give the values of *two* separate quantities x' and y' (which are however connected by the relation $x' = \sqrt{a^2 - y'^2}$) to express the position of a point P on the circle. The above substitution therefore often simplifies solutions of problems.

178. *To find the equation to the straight line joining two points, α and β, on the circle $x^2 + y^2 = a^2$.*

Let the points be P and Q, and let ON be the perpendicular from the origin on the straight line PQ; then ON bisects the angle POQ, and hence

$$\angle XON = \tfrac{1}{2}(\angle XOP + \angle XOQ) = \tfrac{1}{2}(\alpha + \beta).$$

Also $$ON = OP\cos NOP = a\cos\frac{\alpha - \beta}{2}.$$

The equation to PQ is therefore (Art. 53),

$$x\cos\frac{\alpha + \beta}{2} + y\sin\frac{\alpha + \beta}{2} = a\cos\frac{\alpha - \beta}{2}.$$

If we put $\beta = \alpha$ we have, as the equation to the tangent at the point α,

$$x\cos\alpha + y\sin\alpha = a.$$

This may also be deduced from the equation of Art. 150 by putting $x' = a\cos\alpha$ and $y' = a\sin\alpha$.

179. If the equation to the circle be in the more general form

$$(x - h)^2 + (y - k)^2 = a^2, \qquad \text{(Art. 140)},$$

we may express the coordinates of P in the form

$$(h + a\cos\alpha,\ k + a\sin\alpha).$$

For these values satisfy the above equation.

Here α is the angle LCP [Fig. Art. 140].

The equation to the straight line joining the points α and β can be easily shewn to be

$$(x-h)\cos\frac{\alpha+\beta}{2} + (y-k)\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2},$$

and so the tangent at the point α is

$$(x-h)\cos\alpha + (y-k)\sin\alpha = a.$$

***180. Common tangents to two circles.** If O_1 and O_2 be the centres of two circles whose radii are r_1 and r_2, and if one pair of common tangents meet O_1O_2 in T_1 and the other pair meet it in T_2, then, by similar triangles, we have $\frac{O_1T_2}{T_2O_2} = \frac{r_1}{r_2} = \frac{O_1T_1}{O_2T_1}$. The points T_1 and T_2 therefore divide O_1O_2 in the ratio of the radii.

The coordinates of T_1 having been found, the corresponding tangents are straight lines passing through it, such that the perpendiculars on them from O_1 are each equal to r_1. So for the other pair which pass through T_2.

Ex. *Find the four common tangents to the circles*

$$x^2+y^2-22x+4y+100=0, \text{ and } x^2+y^2+22x-4y-100=0.$$

The equations may be written

$$(x-11)^2+(y+2)^2=5^2, \text{ and } (x+11)^2+(y-2)^2=15^2.$$

The centre of the first is the point $(11, -2)$ and its radius is 5.

The centre of the second is the point $(-11, 2)$ and its radius is 15.

Then T_2 is the point dividing internally the line joining the centres in the ratio $5 : 15$ and hence (Art. 22) its coordinates are

$$\frac{15\times 11+5\times(-11)}{15+5} \text{ and } \frac{15\times(-2)+5\times 2}{15+5},$$

that is, T_2 is the point $(\frac{11}{2}, -1)$.

Similarly T_1 is the point dividing this line externally in the ratio $5 : 15$, and hence its coordinates are

$$\frac{15\times 11-5\times(-11)}{15-5} \text{ and } \frac{15\times(-2)-5\times 2}{15-5},$$

that is, T_1 is the point $(22, -4)$.

Let the equation to either of the tangents passing through T_2 be

$$y+1=m\left(x-\tfrac{11}{2}\right)\text{..............................(1).}$$

Then the perpendicular from the point $(11, -2)$ on it is equal to ± 5, and hence

$$\frac{m\left(11-\frac{11}{2}\right)-(-2+1)}{\sqrt{1+m^2}}=\pm 5.$$

On solving, we have $\qquad m=-\frac{24}{7}$ or $\frac{4}{3}$.

The required tangents through T_2 are therefore

$$24x+7y=125, \text{ and } 4x-3y=25.$$

Similarly the equations to the tangents through T_1 are

$$y+4=m(x-22)\text{ ....(2),}$$

where
$$\frac{m(11-22)-(-2+4)}{\sqrt{1+m^2}}=\pm 5.$$

On solving, we have $\qquad m=\frac{7}{24}$ or $-\frac{3}{4}$.

On substitution in (2), the required equations are therefore

$$7x-24y=250 \text{ and } 3x+4y=50.$$

The four common tangents are therefore found.

181. We shall conclude this chapter with some miscellaneous examples on loci.

Ex. 1. *Find the locus of a point P which moves so that its distance from a given point O is always in a given ratio $(n : 1)$ to its distance from another given point A.*

Take O as origin and the direction of OA as the axis of x. Let the distance OA be a, so that A is the point $(a, 0)$.

If (x, y) be the coordinates of any position of P we have

$$OP^2=n^2 . AP^2,$$

i.e. $$x^2+y^2=n^2[(x-a)^2+y^2],$$

i.e. $$(x^2+y^2)(n^2-1)-2an^2x+n^2a^2=0\text{..................(1).}$$

Hence, by Art. 143, the locus of P is a circle.

Let this circle meet the axis of x in the points C and D. Then OC and OD are the roots of the equation obtained by putting y equal to zero in (1).

Hence $$OC=\frac{na}{n+1} \text{ and } OD=\frac{na}{n-1}.$$

We therefore have

$$CA = \frac{a}{n+1} \text{ and } AD = \frac{a}{n-1}.$$

Hence
$$\frac{OC}{CA} = \frac{OD}{AD} = n.$$

The points C and D therefore divide the line OA in the given ratio, and the required circle is on CD as diameter.

Ex. 2. *From any point on one given circle tangents are drawn to another given circle; prove that the locus of the middle point of the chord of contact is a third circle.*

Take the centre of the first circle as origin and let the axis of x pass through the centre of the second circle. Their equations are then

$$x^2 + y^2 = a^2 \quad \text{.......................... (1)},$$

and
$$(x-c)^2 + y^2 = b^2 \quad \text{.......................... (2)},$$

where a and b are the radii, and c the distance between the centres, of the circles.

Any point on (1) is $(a\cos\theta, a\sin\theta)$ where θ is variable. Its chord of contact with respect to (2) is

$$(x-c)(a\cos\theta - c) + ya\sin\theta = b^2 \quad \text{.................. (3)}.$$

The middle point of this chord of contact is the point where it is met by the perpendicular from the centre, *viz.* the point $(c, 0)$.

The equation to this perpendicular is (Art. 70)

$$-(x-c)\,a\sin\theta + (a\cos\theta - c)\,y = 0 \quad \text{.................. (4)}.$$

Any equation deduced from (3) and (4) is satisfied by the coordinates of the point under consideration. If we eliminate θ from them, we shall have an equation always satisfied by the coordinates of the point, whatever be the value of θ. The result will thus be the equation to the required locus.

Solving (3) and (4), we have

$$a\sin\theta = \frac{b^2 y}{y^2 + (x-c)^2},$$

and
$$a\cos\theta - c = \frac{b^2(x-c)}{y^2 + (x-c)^2},$$

so that
$$a\cos\theta = c + \frac{b^2(x-c)}{y^2 + (x-c)^2}.$$

Hence

$$a^2 = a^2\cos^2\theta + a^2\sin^2\theta = c^2 + 2cb^2\frac{x-c}{y^2 + (x-c)^2} + \frac{b^4}{y^2 + (x-c)^2}.$$

The required locus is therefore

$$(a^2 - c^2)\left[y^2 + (x-c)^2\right] = 2cb^2(x-c) + b^4.$$

This is a circle and its centre and radius are easily found.

Ex. 3. *Find the locus of a point P which is such that its polar with respect to one circle touches a second circle.*

Taking the notation of the last article, the equations to the two circles are

$$x^2+y^2=a^2 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and

$$(x-c)^2+y^2=b^2 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Let (h, k) be the coordinates of any position of P. Its polar with respect to (1) is

$$xh+yk=a^2 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Also any tangent to (2) has its equation of the form (Art. 179)

$$(x-c)\cos\theta+y\sin\theta=b \quad \ldots\ldots\ldots\ldots\ldots (4).$$

If then (3) be a tangent to (2) it must be of the form (4).

Therefore
$$\frac{\cos\theta}{h}=\frac{\sin\theta}{k}=\frac{c\cos\theta+b}{a^2}.$$

These equations give

$$\cos\theta\,(a^2-ch)=bh, \text{ and } \sin\theta\,(a^2-ch)=bk.$$

Squaring and adding, we have

$$(a^2-ch)^2=b^2(h^2+k^2) \quad \ldots\ldots\ldots\ldots\ldots (5).$$

The locus of the point (h, k) is therefore the curve

$$b^2(x^2+y^2)=(a^2-cx)^2.$$

Aliter. The condition that (3) may touch (2) may be otherwise found.

For, as in Art. 153, the straight line (3) meets the circle (2) in the points whose abscissæ are given by the equation

$$k^2(x-c)^2+(a^2-hx)^2=b^2k^2,$$

i.e.
$$x^2(h^2+k^2)-2x(ck^2+a^2h)+(k^2c^2+a^4-b^2k^2)=0.$$

The line (3) will therefore touch (2) if

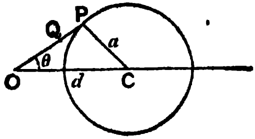
$$(ck^2+a^2h)^2=(h^2+k^2)(k^2c^2+a^4-b^2k^2),$$

i.e. if
$$b^2(h^2+k^2)=(ch-a^2)^2,$$

which is equation (5).

Ex. 4. *O is a fixed point and P any point on a given circle; OP is joined and on it a point Q is taken so that OP . OQ = a constant quantity k^2; prove that the locus of Q is a circle which becomes a straight line when O lies on the original circle.*

Let O be taken as pole and the line through the centre C as the initial line. Let $OC=d$, and let the radius of the circle be a.



The equation to the circle is then

$a^2=r^2+d^2-2rd\cos\theta$, (Art. 171),

where $OP=r$ and $\angle POC=\theta$.

Let OQ be ρ, so that, by the given condition, we have $r\rho=k^2$ and hence $r=\frac{k^2}{\rho}$.

Substituting this value in the equation to the circle, we have

$$a^2=\frac{k^4}{\rho^2}+d^2-2\frac{k^2d}{\rho}\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

so that the equation to the locus of Q is

$$r^2-2\frac{k^2d}{d^2-a^2}r\cos\theta=-\frac{k^4}{d^2-a^2}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But the equation to a circle, whose radius is a' and whose centre is on the initial line at a distance d', is

$$r^2-2rd'\cos\theta=a'^2-d'^2\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Comparing (1) and (2), we see that the required locus is a circle, such that

$$d'=\frac{k^2d}{d^2-a^2} \text{ and } a'^2-d'^2=-\frac{k^4}{d^2-a^2}.$$

Hence $$a'^2=\frac{k^4}{d^2-a^2}\left[\frac{d^2}{d^2-a^2}-1\right]=\frac{k^4a^2}{(d^2-a^2)^2}.$$

The required locus is therefore a circle, of radius $\frac{k^2a}{d^2-a^2}$, whose centre is on the same line as the original centre at a distance $\frac{k^2d}{d^2-a^2}$ from the fixed point.

When O lies on the original circle the distance d is equal to a, and the equation (1) becomes $k^2=2dr\cos\theta$, *i.e.*, in Cartesian coordinates,

$$x=\frac{k^2}{2d}.$$

In this case the required locus is a straight line perpendicular to OC.

When a second curve is obtained from a given curve by the above geometrical process, the second curve is said to be the **inverse** of the first curve and the fixed point O is called the centre of inversion.

The inverse of a circle is therefore a circle or a straight line according as the centre of inversion is not, or is, on the circumference of the original circle.

Ex. 5. *PQ is a straight line drawn through O, one of the common points of two circles, and meets them again in P and Q; find the locus of the point S which bisects the line PQ.*

Take O as the origin, let the radii of the two circles be R and R', and let the lines joining their centres to O make angles α and α' with the initial line.

The equations to the two circles are therefore, {Art. 172 (2)},

$$r = 2R\cos(\theta - \alpha), \text{ and } r = 2R'\cos(\theta - \alpha').$$

Hence, if S be the middle point of PQ, we have

$$2OS = OP + OQ = 2R\cos(\theta - \alpha) + 2R'\cos(\theta - \alpha').$$

The locus of the point S is therefore

$$r = R\cos(\theta - \alpha) + R'\cos(\theta - \alpha')$$

$$= (R\cos\alpha + R'\cos\alpha')\cos\theta + (R\sin\alpha + R'\sin\alpha')\sin\theta$$

$$= 2R''\cos(\theta - \alpha'') \quad \ldots\ldots\ldots\ldots (1),$$

where $\qquad 2R''\cos\alpha'' = R\cos\alpha + R'\cos\alpha',$

and $\qquad 2R''\sin\alpha'' = R\sin\alpha + R'\sin\alpha'.$

Hence $\qquad R'' = \frac{1}{2}\sqrt{R^2 + R'^2 + 2RR'\cos(\alpha - \alpha')},$

and $\qquad \tan\alpha'' = \dfrac{R\sin\alpha + R'\sin\alpha'}{R\cos\alpha + R'\cos\alpha'}.$

From (1) the locus of S is a circle, whose radius is R'', which passes through the origin O and is such that the line joining O to its centre is inclined at an angle α'' to the initial line.

EXAMPLES. XXII.

1. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; prove that it always lies on a circle.

2. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant; prove that its locus is a circle.

3. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant; prove that its locus is a circle.

4. Find the locus of a point which moves so that the square of the tangent drawn from it to the circle $x^2 + y^2 = a^2$ is equal to c times its distance from the straight line $lx + my + n = 0$.

5. Find the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle.

6. Find the locus of the vertex of a triangle, given (1) its base and the sum of the squares of its sides, (2) its base and the sum of m times the square of one side and n times the square of the other.

7. A point moves so that the sum of the squares of its distances from n fixed points is given. Prove that its locus is a circle.

8. Whatever be the value of a, prove that the locus of the intersection of the straight lines

$$x \cos a + y \sin a = a \text{ and } x \sin a - y \cos a = b$$

is a circle.

9. From a point P on a circle perpendiculars PM and PN are drawn to two radii of the circle which are not at right angles; find the locus of the middle point of MN.

10. Tangents are drawn to a circle from a point which always lies on a given line; prove that the locus of the middle point of the chord of contact is another circle.

11. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which pass through the fixed point (h, k).

12. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the point $(c, 0)$.

13. O is a fixed point and P any point on a fixed circle; on OP is taken a point Q such that OQ is in a constant ratio to OP; prove that the locus of Q is a circle.

14. O is a fixed point and P any point on a given straight line; OP is joined and on it is taken a point Q such that $OP \cdot OQ = k^2$; prove that the locus of Q, *i.e.* the inverse of the given straight line with respect to O, is a circle which passes through O.

15. One vertex of a triangle of given species is fixed, and another moves along the circumference of a fixed circle; prove that the locus of the remaining vertex is a circle and find its radius.

16. O is any point in the plane of a circle, and OP_1P_2 any chord of the circle which passes through O and meets the circle in P_1 and P_2. On this chord is taken a point Q such that OQ is equal to (1) the arithmetic, (2) the geometric, and (3) the harmonic mean between OP_1 and OP_2; in each case find the equation to the locus of Q.

17. Find the locus of the point of intersection of the tangent to a given circle and the perpendicular let fall on this tangent from a fixed point on the circle.

18. A circle touches the axis of x and cuts off a constant length $2l$ from the axis of y; prove that the equation of the locus of its centre is $y^2 - x^2 = l^2 \operatorname{cosec}^2 \omega$, the axes being inclined at an angle ω.

19. A straight line moves so that the product of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of the perpendiculars from each of these points upon the straight line is a circle, the same for each.

20. O is a fixed point and AP and BQ are two fixed parallel straight lines; BOA is perpendicular to both and POQ is a right angle. Prove that the locus of the foot of the perpendicular drawn from O upon PQ is the circle on AB as diameter.

21. Two rods, of lengths a and b, slide along the axes, which are rectangular, in such a manner that their ends are always concyclic; prove that the locus of the centre of the circle passing through these ends is the curve $4(x^2-y^2)=a^2-b^2$.

22. Shew that the locus of a point, which is such that the tangents from it to two given concentric circles are inversely as the radii, is a concentric circle, the square of whose radius is equal to the sum of the squares of the radii of the given circles.

23. Shew that if the length of the tangent from a point P to the circle $x^2+y^2=a^2$ be four times the length of the tangent from it to the circle $(x-a)^2+y^2=a^2$, then P lies on the circle

$$15x^2+15y^2-32ax+a^2=0.$$

Prove also that these three circles pass through two points and that the distance between the centres of the first and third circles is sixteen times the distance between the centres of the second and third circles.

24. Find the locus of the foot of the perpendicular let fall from the origin upon any chord of the circle $x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.

Find also the locus of the middle points of these chords.

25. Through a fixed point O are drawn two straight lines OPQ and ORS to meet a circle in P and Q, and R and S, respectively. Prove that the locus of the point of intersection of PS and QR, as also that of the point of intersection of PR and QS, is the polar of O with respect to the circle.

26. A, B, C, and D are four points in a straight line; prove that the locus of a point P, such that the angles APB and CPD are equal, is a circle.

27. The polar of P with respect to the circle $x^2+y^2=a^2$ touches the circle $(x-\alpha)^2+(y-\beta)^2=b^2$; prove that its locus is the curve given by the equation $(\alpha x+\beta y-a^2)^2=b^2(x^2+y^2)$.

28. A tangent is drawn to the circle $(x-a)^2+y^2=b^2$ and a perpendicular tangent to the circle $(x+a)^2+y^2=c^2$; find the locus of their point of intersection, and prove that the bisector of the angle between them always touches one or other of two fixed circles.

29. In any circle prove that the perpendicular from any point of it on the line joining the points of contact of two tangents is a mean proportional between the perpendiculars from the point upon the two tangents.

30. From any point on the circle

$$x^2+y^2+2gx+2fy+c=0$$

tangents are drawn to the circle

$$x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0;$$

prove that the angle between them is 2α.

31. The angular points of a triangle are the points

$$(a\cos\alpha,\ a\sin\alpha),\ (a\cos\beta,\ a\sin\beta),\ \text{and}\ (a\cos\gamma,\ a\sin\gamma);$$

prove that the coordinates of the orthocentre of the triangle are

$$a(\cos\alpha+\cos\beta+\cos\gamma)\ \text{and}\ a(\sin\alpha+\sin\beta+\sin\gamma).$$

Hence prove that if A, B, C, and D be four points on a circle the orthocentres of the four triangles ABC, BCD, CDA, and DAB lie on a circle.

32. A variable circle passes through the point of intersection O of any two straight lines and cuts off from them portions OP and OQ such that $m\,.\,OP+n\,.\,OQ$ is equal to unity; prove that this circle always passes through a fixed point.

33. Find the length of the common chord of the circles, whose equations are $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$, and prove that the equation to the circle whose diameter is this common chord is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

34. Prove that the length of the common chord of the two circles whose equations are

$$(x-a)^2+(y-b)^2=c^2\ \text{and}\ (x-b)^2+(y-a)^2=c^2$$

is $$\sqrt{4c^2-2(a-b)^2}.$$

Hence find the condition that the two circles may touch.

35. Find the length of the common chord of the circles

$$x^2+y^2-2ax-4ay-4a^2=0\ \text{and}\ x^2+y^2-3ax+4ay=0.$$

Find also the equations of the common tangents and shew that the length of each is $4a$.

36. Find the equations to the common tangents of the circles

(1) $x^2+y^2-2x-6y+9=0$ and $x^2+y^2+6x-2y+1=0$,

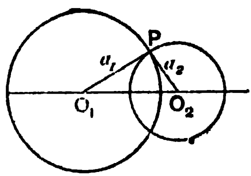
(2) $x^2+y^2=c^2$ and $(x-a)^2+y^2=b^2$.

CHAPTER IX.

SYSTEMS OF CIRCLES.

[This chapter may be omitted by the student on a first reading of the subject.]

182. Orthogonal Circles. Def. Two circles are said to intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii O_1P and O_2P, which are perpendicular to the tangents at P, must also be at right angles.

Hence $$O_1O_2^2 = O_1P^2 + O_2P^2,$$

i.e. the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent from O_2 to the other circle is equal to the radius a_2, *i.e.* if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.

Either of these two conditions will determine whether the circles are orthogonal.

The centres of the circles

$$x^2+y^2+2gx+2fy+c=0 \text{ and } x^2+y^2+2g'x+2f'y+c'=0,$$

are the points $(-g, -f)$ and $(-g', -f')$; also the squares of their radii are g^2+f^2-c and $g'^2+f'^2-c'$.

They therefore cut orthogonally if

$$(-g+g')^2+(-f+f')^2=g^2+f^2-c+g'^2+f'^2-c',$$

i.e. if $$2gg'+2ff'=c+c'.$$

183. Radical Axis. Def. The radical axis of two circles is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal.

Let the equations to the circles be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots\ldots (1),$$

and $$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots\ldots (2),$$

and let (x_1, y_1) be any point such that the tangents from it to these circles are equal.

By Art. 168, we have

$$x_1^2+y_1^2+2gx_1+2fy_1+c=x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1,$$

i.e. $$2x_1(g-g_1)+2y_1(f-f_1)+c-c_1=0.$$

But this is the condition that the point (x_1, y_1) should lie on the locus

$$2x(g-g_1)+2y(f-f_1)+c-c_1=0 \quad \ldots\ldots (3).$$

This is therefore the equation to the radical axis, and it is clearly a straight line.

It is easily seen that the radical axis is perpendicular to the line joining the centres of the circles. For these centres are the points $(-g, -f)$ and $(-g_1, -f_1)$. The "m" of the line joining them is therefore $\frac{-f_1-(-f)}{-g_1-(-g)}$,

i.e. $\frac{f-f_1}{g-g_1}$.

The "m" of the line (3) is $-\frac{g-g_1}{f-f_1}$.

The product of these two "m's" is -1.

Hence, by Art. 69, the radical axis and the line joining the centres are perpendicular.

184. A geometrical construction can be given for the radical axis of two circles.

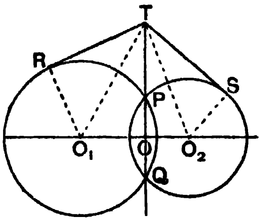


Fig. 1.

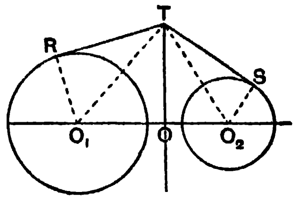


Fig. 2.

If the circles intersect in real points, P and Q, as in Fig. 1, the radical axis is clearly the straight line PQ. For if T be any point on PQ and TR and TS be the tangents from it to the circles we have

$$TR^2 = TP \,.\, TQ = TS^2.$$

If they do not intersect in real points, as in the second figure, let their radii be a_1 and a_2, and let T be a point such that the tangents TR and TS are equal in length.

Draw TO perpendicular to O_1O_2.

Since $$TR^2 = TS^2,$$

we have $$TO_1^2 - O_1R^2 = TO_2^2 - O_2S^2,$$

i.e. $$TO^2 + O_1O^2 - a_1^2 = TO^2 + OO_2^2 - a_2^2,$$

i.e. $$O_1O^2 - OO_2^2 = a_1^2 - a_2^2,$$

i.e. $$(O_1O - OO_2)(O_1O + OO_2) = a_1^2 - a_2^2,$$

i.e. $$O_1O - OO_2 = \frac{a_1^2 - a_2^2}{O_1O_2} = \text{a constant quantity.}$$

Hence O is a fixed point, since it divides the fixed straight line O_1O_2 into parts whose difference is constant.

Therefore, since O_1OT is a right angle, the locus of T, *i.e.* the radical axis, is a fixed straight line perpendicular to the line joining the centres.

185. If the equations to the circles in Art. 183 be written in the form $S=0$ and $S'=0$, the equation (3) to the radical axis may be written $S-S'=0$, and therefore the radical axis passes through the common points, real or imaginary, of the circles $S=0$ and $S'=0$.

In the last article we saw that this was true geometrically for the case in which the circles meet in real points.

When the circles do not geometrically intersect, as in Fig. 2, we must then look upon the straight line TO as passing through the imaginary points of intersection of the two circles.

186. *The radical axes of three circles, taken in pairs, meet in a point.*

Let the equations to the three circles be

$$S=0 \text{ .... } (1),$$

$$S'=0 \text{ .... } (2),$$

and

$$S''=0 \text{ .... } (3).$$

The radical axis of the circles (1) and (2) is the straight line

$$S-S'=0 \text{ .... } (4).$$

The radical axis of (2) and (3) is the straight line

$$S'-S''=0 \text{ .... } (5).$$

If we add equation (5) to equation (4) we shall have the equation of a straight line through their points of intersection.

Hence $$S-S''=0 \text{ .... } (6)$$

is a straight line through the intersection of (4) and (5).

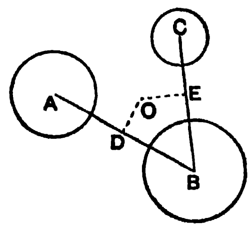
But (6) is the radical axis of the circles (3) and (1).

Hence the three radical axes of the three circles, taken in pairs, meet in a point.

This point is called the **Radical Centre** of the three circles.

This may also be easily proved geometrically. For let the three circles be called A, B, and C, and let the radical axis of A and B and that of B and C meet in a point O.

By the definition of the radical axis, the tangent from O to the circle A = the tangent from O to the circle B, and the tangent from O to the circle B = tangent from it to the circle C.



Hence the tangent from O to the circle A = the tangent from it to the circle C, *i.e.* O is also a point on the radical axis of the circles A and C.

187. *If $S=0$ and $S'=0$ be the equations of two circles, the equation of any circle through their points of intersection is $S=\lambda S'$. Also the equation to any circle, such that the radical axis of it and $S=0$ is $u=0$, is $S+\lambda u=0$.*

For wherever $S=0$ and $S'=0$ are *both* satisfied the equation $S=\lambda S'$ is clearly satisfied, so that $S=\lambda S'$ is some locus through the intersections of $S=0$ and $S'=0$.

Also in both S and S' the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero. The same statement is therefore true for the equation $S-\lambda S'$. Hence the proposition.

Again, since u is only of the first degree, therefore in $S+\lambda u$ the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero, so that $S+\lambda u=0$ is clearly a circle. Also it passes through the intersections of $S=0$ and $u=0$.

EXAMPLES. XXIII.

Prove that the following pairs of circles intersect orthogonally:

1. $x^2+y^2-2ax+c=0$ and $x^2+y^2+2by-c=0$.

2. $x^2+y^2-2ax+2by+c=0$ and $x^2+y^2+2bx+2ay-c=0$.

3. Find the equation to the circle which passes through the origin and cuts orthogonally each of the circles

$$x^2+y^2-6x+8=0 \text{ and } x^2+y^2-2x-2y=7.$$

Find the radical axis of the pairs of circles

4. $x^2+y^2=144$ and $x^2+y^2-15x+11y=0$.

5. $x^2+y^2-3x-4y+5=0$ and $3x^2+3y^2-7x+8y+11=0$.

6. $x^2+y^2-xy+6x-7y+8=0$ and $x^2+y^2-xy-4=0$, the axes being inclined at 120°.

Find the radical centre of the sets of circles

7. $x^2+y^2+x+2y+3=0$, $\quad x^2+y^2+2x+4y+5=0$,

and $\quad x^2+y^2-7x-8y-9=0$.

8. $(x-2)^2+(y-3)^2=36$, $\quad (x+3)^2+(y+2)^2=49$,

and $\quad (x-4)^2+(y+5)^2=64$.

9. Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles, and the distance between their centres.

10. Prove that a common tangent to two circles is bisected by the radical axis. [Hence, by joining the middle points of any two of the common tangents, we have a construction for the radical axis.]

11. Find the general equation of all circles any pair of which have the same radical axis as the circles

$$x^2+y^2=4 \text{ and } x^2+y^2+2x+4y=6.$$

12. Find the equations to the straight lines joining the origin to the points of intersection of

$$x^2+y^2-4x-2y=4 \text{ and } x^2+y^2-2x-4y-4=0.$$

13. The polars of a point P with respect to two fixed circles meet in the point Q. Prove that the circle on PQ as diameter passes through two fixed points, and cuts both the given circles at right angles.

14. Prove that the two circles, which pass through the two points $(0, a)$ and $(0, -a)$ and touch the straight line $y=mx+c$, will cut orthogonally if $c^2=a^2(2+m^2)$.

15. Find the locus of the centre of the circle which cuts two given circles orthogonally.

16. If two circles cut orthogonally, prove that the polar of any point P on the first circle with respect to the second passes through the other end of the diameter of the first circle which goes through P.

Hence, (by considering the orthogonal circle of three circles as the locus of a point such that its polars with respect to the circles meet in a point) prove that the orthogonal circle of three circles, given by the general equation is

$$\begin{vmatrix} x+g_1, & y+f_1, & g_1x+f_1y+c_1 \\ x+g_2, & y+f_2, & g_2x+f_2y+c_2 \\ x+g_3, & y+f_3, & g_3x+f_3y+c_3 \end{vmatrix}=0.$$

188. Coaxal Circles. Def. A system of circles is said to be coaxal when they have a common radical axis, *i.e.* when the radical axis of each pair of circles of the system is the same.

To find the equation of a system of coaxal circles.

Since, by Art. 183, the radical axis of any pair of the circles is perpendicular to the line joining their centres, it follows that the centres of all the circles of a coaxal system must lie on a straight line which is perpendicular to the radical axis.

Take the line of centres as the axis of x and the radical axis as the axis of y (Figs. I. and II., Art. 190), so that O is the origin.

The equation to any circle with its centre on the axis of x is

$$x^2 + y^2 - 2gx + c = 0 \quad\quad\text{.................(1).}$$

Any point on the radical axis is $(0, y_1)$.

The square on the tangent from it to the circle (1) is, by Art. 168, $y_1^2 + c$.

Since this quantity is to be the same for all circles of the system it follows that c is the same for all such circles; the different circles are therefore obtained by giving different values to g in the equation (1).

The intersections of (1) with the radical axis are then obtained by putting $x = 0$ in equation (1), and we have

$$y = \pm\sqrt{-c}.$$

If c be negative, we have two real points of intersection as in Fig. I. of Art. 190. In such cases the circles are said to be of the Intersecting Species.

If c be positive, we have two imaginary points of intersection as in Fig. II.

189. Limiting points of a coaxal system.

The equation (1) of the previous article which gives any circle of the system may be written in the form

$$(x-g)^2 + y^2 = g^2 - c = [\sqrt{g^2 - c}]^2.$$

It therefore represents a circle whose centre is the point $(g, 0)$ and whose radius is $\sqrt{g^2-c}$.

This radius vanishes, *i.e.* the circle becomes a point-circle, when $g^2=c$, *i.e.* when $g=\pm\sqrt{c}$.

Hence at the particular points $(\pm\sqrt{c}, 0)$ we have point-circles which belong to the system. These point-circles are called the Limiting Points of the system.

If c be negative, these points are imaginary.

But it was shown in the last article that when c is negative the circles intersect in real points as in Fig. I., Art. 190.

If c be positive, the limiting points L_1 and L_2 (Fig. II.) are real, and in this case the circles intersect in imaginary points.

The limiting points are therefore real or imaginary according as the circles of the system intersect in imaginary or real points.

190. Orthogonal circles of a coaxal system.

Let T be *any* point on the common radical axis of a system of coaxal circles, and let TR be the tangent from it to any circle of the system.

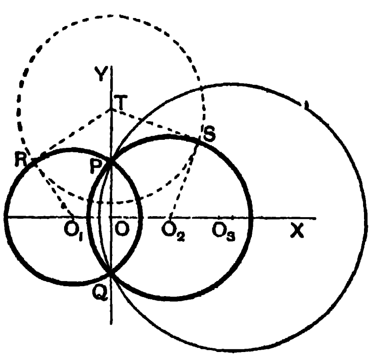


Fig. I.

Then a circle, whose centre is T and whose radius is TR, will cut each circle of the coaxal system orthogonally.

[For the radius TR of this circle is at right angles to the radius O_1R, and so for its intersection with *any* other circle of the system.]

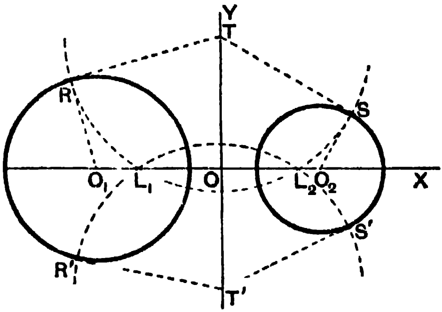


Fig. II.

Hence the limiting points (being point-*circles* of the system) are on this orthogonal circle.

The limiting points are therefore the intersections with the line of centres of *any* circle whose centre is on the common radical axis and whose radius is the tangent from it to any of the circles of the system.

Since, in Fig. I., the limiting points are imaginary these orthogonal circles do not meet the line of centres in real points.

In Fig. II. they pass through the limiting points L_1 and L_2.

These orthogonal circles (since they all pass through two points, real or imaginary) are therefore a coaxal system.

Also if the original circles, as in Fig. I., intersect in real points, the orthogonal circles intersect in imaginary points; in Fig. II. the original circles intersect in imaginary points, and the orthogonal circles in real points.

We therefore have the following theorem:

A set of coaxal circles can be cut orthogonally by another set of coaxal circles, the centres of each set lying on the radical axis of the other set; also one set is of the limiting-point species and the other set of the other species.

191. Without reference to the limiting points of the original system, it may be easily found whether or not the orthogonal circles meet the original line of centres.

For the circle, whose centre is T and whose radius is TR, meets or does not meet the line O_1O_2 according as TR^2 is $>$ or $<$ TO^2,

i.e. according as $\quad TO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as $\quad TO^2 + OO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as $\quad OO_1$ is $\gtrless O_1R$,

i.e. according as the radical axis is without, or within, each of the circles of the original system.

192. In the next article the above results will be proved analytically.

To find the equation to any circle which cuts two given circles orthogonally.

Take the radical axis of the two circles as the axis of y, so that their equations may be written in the form

$$x^2 + y^2 - 2gx + c = 0 \qquad \ldots\ldots\ldots(1),$$

and

$$x^2 + y^2 - 2g_1x + c = 0 \qquad \ldots\ldots\ldots(2),$$

the quantity c being the same for each.

Let the equation to any circle which cuts them orthogonally be

$$(x - A)^2 + (y - B)^2 = R^2 \ldots\ldots\ldots\ldots\ldots(3).$$

The equation (1) can be written in the form

$$(x - g)^2 + y^2 = [\sqrt{g^2 - c}]^2 \ldots\ldots\ldots\ldots\ldots(4).$$

The circles (3) and (4) cut orthogonally if the square of the distance between their centres is equal to the sum of the squares of their radii,

i.e. if $\qquad (A - g)^2 + B^2 = R^2 + [\sqrt{g^2 - c}]^2,$

i.e. if $\qquad A^2 + B^2 - 2Ag = R^2 - c \ldots\ldots\ldots\ldots(5).$

Similarly, (3) will cut (2) orthogonally if

$$A^2 + B^2 - 2Ag_1 = R^2 - c \ldots\ldots\ldots\ldots\ldots(6).$$

Subtracting (6) from (5), we have $A(g - g_1) = 0$.

Hence $A = 0$, and $R^2 = B^2 + c$.

Substituting these values in (3), the equation to the required orthogonal circle is

$$x^2 + y^2 - 2By - c = 0 \quad\text{.................(7),}$$

where B is any quantity whatever.

Whatever be the value of B the equation (7) represents a circle whose centre is on the axis of y and which passes through the points $(\pm\sqrt{c}, 0)$.

But the latter points are the limiting points of the coaxal system to which the two circles belong. [Art. 189.]

Hence any pair of circles belonging to a coaxal system is cut at right angles by any circle of another coaxal system; also the centres of the circles of the latter system lie on the common radical axis of the original system, and all the circles of the latter system pass through the limiting points (real or imaginary) of the first system.

Also the centre of the circle (7) is the point $(0, B)$ and its radius is $\sqrt{B^2 + c}$.

The square of the tangent drawn from $(0, B)$ to the circle (1) $= B^2 + c$ (by Art. 168).

Hence the radius of any circle of the second system is equal to the length of the tangent drawn from its centre to any circle of the first system.

193. The equation to the system of circles which cut a given coaxal system orthogonally may also be obtained by using the result of Art. 182.

For any circle of the coaxal system is, by Art. 188, given by

$$x^2 + y^2 - 2gx + c = 0 \quad\text{.................(1),}$$

where c is the same for all circles.

Any point on the radical axis is $(0, y')$.

The square on the tangent drawn from it to (1) is therefore $y'^2 + c$.

The equation to *any* circle cutting (1) orthogonally is therefore

$$x^2 + (y - y')^2 = y'^2 + c,$$

i.e.

$$x^2 + y^2 - 2yy' - c = 0.$$

Whatever be the value of y' this circle passes through the points $(\pm\sqrt{c}, 0)$, *i.e.* through the limiting points of the system of circles given by (1).

194. We can now deduce an easy construction for the circle that cuts any three circles orthogonally.

Consider the three circles in the figure of Art. 186.

By Art. 192 any circle cutting A and B orthogonally has its centre on their common radical axis, *i.e.* on the straight line OD.

Similarly any circle cutting B and C orthogonally has its centre on the radical axis OE.

Any circle cutting all three circles orthogonally must therefore have its centre at the intersection of OD and OE, *i.e.* at the radical centre O. Also its radius must be the length of the tangent drawn from the radical centre to any one of the three circles.

Ex. *Find the equation to the circle which cuts orthogonally each of the three circles*

$$x^2+y^2+2x+17y+4=0 \quad\text{.................. (1)},$$
$$x^2+y^2+7x+6y+11=0 \quad\text{.................. (2)},$$
$$x^2+y^2-x+22y+3=0 \quad\text{.................. (3)}.$$

The radical axis of (1) and (2) is

$$5x-11y+7=0.$$

The radical axis of (2) and (3) is

$$8x-16y+8=0.$$

These two straight lines meet in the point (3, 2) which is therefore the radical centre.

The square of the length of the tangent from the point (3, 2) to each of the given circles $=57$.

The required equation is therefore $(x-3)^2+(y-2)^2=57$,

i.e. $$x^2+y^2-6x-4y-44=0.$$

195. Ex. *Find the locus of a point which moves so that the length of the tangent drawn from it to one given circle is λ times the length of the tangent from it to another given circle.*

As in Art. 188 take as axes of x and y the line joining the centres of the two circles and the radical axis. The equations to the two circles are therefore

$$x^2+y^2-2g_1x+c=0 \quad\text{.................. (1)},$$

and $$x^2+y^2-2g_2x+c=0 \quad\text{.................. (2)}.$$

Let (h, k) be a point such that the length of the tangent from it to (1) is always λ times the length of the tangent from it to (2).

Then $$h^2+k^2-2g_1h+c=\lambda^2[h^2+k^2-2g_2h+c].$$

Hence (h, k) always lies on the circle

$$x^2+y^2-2x\frac{g_2\lambda^2-g_1}{\lambda^2-1}+c=0 \quad\text{.....................(3).}$$

This circle is clearly a circle of the coaxal system to which (1) and (2) belong.

Again, the centre of (1) is the point $(g_1, 0)$, the centre of (2) is $(g_2, 0)$, whilst the centre of (3) is $\left(\frac{g_2\lambda^2-g_1}{\lambda^2-1}, 0\right)$.

Hence, if these three centres be called O_1, O_2, and O_3, we have

$$O_1O_3=\frac{g_2\lambda^2-g_1}{\lambda^2-1}-g_1=\frac{\lambda^2}{\lambda^2-1}(g_2-g_1),$$

and $$O_2O_3=\frac{g_2\lambda^2-g_1}{\lambda^2-1}-g_2=\frac{1}{\lambda^2-1}(g_2-g_1),$$

so that $O_1O_3 : O_2O_3 :: \lambda^2 : 1$.

The required locus is therefore a circle coaxal with the two given circles and whose centre divides externally, in the ratio $\lambda^2 : 1$, the line joining the centres of the two given circles.

EXAMPLES. XXIV.

1. Prove that a common tangent to two circles of a coaxal system subtends a right angle at either limiting point of the system.

2. Prove that the polar of a limiting point of a coaxal system with respect to any circle of the system is the same for all circles of the system.

3. Prove that the polars of any point with respect to a system of coaxal circles all pass through a fixed point, and that the two points are equidistant from the radical axis and subtend a right angle at a limiting point of the system. If the first point be one limiting point of the system prove that the second point is the other limiting point.

4. A fixed circle is cut by a series of circles all of which pass through two given points; prove that the straight line joining the intersections of the fixed circle with any circle of the system always passes through a fixed point.

5. Prove that tangents drawn from any point of a fixed circle of a coaxal system to two other fixed circles of the system are in a constant ratio.

6. Prove that a system of coaxal circles inverts with respect to either limiting point into a system of concentric circles and find the position of the common centre.

7. A straight line is drawn touching one of a system of coaxal circles in P and cutting another in Q and R. Shew that PQ and PR subtend equal or supplementary angles at one of the limiting points of the system.

8. Find the locus of the point of contact of parallel tangents which are drawn to each of a series of coaxal circles.

9. Prove that the circle of similitude of the two circles

$$x^2+y^2-2kx+\delta=0 \text{ and } x^2+y^2-2k'x+\delta=0$$

(*i.e.* the locus of the points at which the two circles subtend the same angle) is the coaxal circle

$$x^2+y^2-2\frac{kk'+\delta}{k+k'}x+\delta=0.$$

10. From the preceding question shew that the centres of similitude (*i.e.* the points in which the common tangents to two circles meet the line of centres) divide the line joining the centres internally and externally in the ratio of the radii.

11. If $x+y\sqrt{-1}=\tan(u+v\sqrt{-1})$, where x, y, u, and v are all real, prove that the curves $u=$ constant give a family of coaxal circles passing through the points $(0, \pm 1)$, and that the curves $v=$ constant give a system of circles cutting the first system orthogonally.

12. Find the equation to the circle which cuts orthogonally each of the circles

$$x^2+y^2+2gx+c=0, \qquad x^2+y^2+2g'x+c=0,$$

and

$$x^2+y^2+2hx+2ky+a=0.$$

13. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2=a^2, \quad (x-c)^2+y^2=a^2, \quad \text{and} \quad x^2+(y-b)^2=a^2.$$

14. Find the equation to the circle cutting orthogonally the three circles

$$x^2+y^2-2x+3y-7=0, \qquad x^2+y^2+5x-5y+9=0,$$

and

$$x^2+y^2+7x-9y+29=0.$$

15. Shew that the equation to the circle cutting orthogonally the circles

$$(x-a)^2+(y-b)^2=b^2, \qquad (x-b)^2+(y-a)^2=a^2,$$

and

$$(x-a-b-c)^2+y^2=ab+c^2,$$

is

$$x^2+y^2-2x(a+b)-y(a+b)+a^2+3ab+b^2=0.$$

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ANSWERS.

I. (Pages 14, 15.)

1. 5. **2.** 13. **3.** $3\sqrt{7}$. **4.** $\sqrt{a^2+b^2}$.

5. $\sqrt{a^2+2b^2+c^2-2ab-2bc}$. **6.** $2a\sin\frac{\alpha-\beta}{2}$.

7. $a(m_1-m_2)\sqrt{(m_1+m_2)^2+4}$. **9.** $3\pm 2\sqrt{15}$.

15. $(\frac{10}{7}, \frac{33}{7})$. **16.** $(-2, -9)$. **17.** $(1, -\frac{1}{5})$; $(-11, 16)$.

18. $(-5\frac{1}{12}, 2\frac{5}{12})$; $(-20\frac{1}{2}, 34\frac{1}{2})$. **19.** $(-\frac{1}{3}, 0)$; $(-\frac{5}{3}, 2)$.

20. $(-\frac{5}{2}, \frac{3}{2})$; $(1, 1)$; $(\frac{3}{2}, -\frac{5}{2})$.

21. $\left(\frac{a^2+b^2}{a+b}, \frac{a^2+2ab-b^2}{a+b}\right)$; $\left(\frac{a^2-2ab-b^2}{a-b}, \frac{a^2+b^2}{a-b}\right)$.

22. $\left(\frac{kx_1+lx_2+mx_3}{k+l+m}, \frac{ky_1+ly_2+my_3}{k+l+m}\right)$.

II. (Pages 18, 19.)

1. 10. **2.** 1. **3.** 29. **4.** $2ac$.

5. a^2. **6.** $2ab\sin\frac{\phi_2-\phi_3}{2}\sin\frac{\phi_3-\phi_1}{2}\sin\frac{\phi_1-\phi_2}{2}$.

7. $a^3(m_2-m_3)(m_3-m_1)(m_1-m_2)$.

8. $\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2)$.

9. $\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2)\div m_1m_2m_3$.

13. $20\frac{1}{2}$. **14.** 96.

III. (Pages 22, 23.)

12. $2\sqrt{5}$. **13.** $\sqrt{79}$. **14.** $\sqrt{7}a$. **16.** $\frac{1}{4}(8-3\sqrt{3})$.

17. $\frac{7\sqrt{3}}{2}$. **18.** $\frac{1}{4}a^2\sqrt{3}$. **25.** $r^2=a^2$. **26.** $\theta=\alpha$.

27. $r=2a\cos\theta$. **28.** $r\cos 2\theta=2a\sin\theta$. **29.** $r\cos\theta=2a\sin^2\theta$.

30. $r^2=a^2\cos 2\theta$. **31.** $x^2+y^2=a^2$. **32.** $y=mx$.

33. $x^2+y^2=ax$. **34.** $(x^2+y^2)^3=4a^2x^2y^2$.

35. $(x^2+y^2)^2=a^2(x^2-y^2)$. **36.** $xy=a^2$. **37.** $x^2-y^2=a^2$.

38. $y^2+4ax=4a^2$. **39.** $4(x^2+y^2)(x^2+y^2+ax)=a^2y^2$.

40. $x^3-3xy^2+3x^2y-y^3=5kxy$.

IV. (Page 30.)

8. $2ax+k^2=0$. **9.** $(n^2-1)(x^2+y^2+a^2)+2ax(n^2+1)=0$.
10. $4x^2(c^2-4a^2)+4c^2y^2=c^2(c^2-4a^2)$. **11.** $(6a-2c)x=a^2-c^2$.
12. $y^2-4y-2x+5=0$. **13.** $4y+2x+3=0$. **14.** $x+y=7$.
15. $y=x$. **16.** $y=3x$. **17.** $15x^2-y^2+2ax=a^2$.
18. $x^2+y^2=3$. **19.** $x^2+y^2=4y$.
20. $8x^2+8y^2+6x-36y+27=0$. **21.** $x^2=3y^2$.
22. $x^2+2ay=a^2$.
23. (1) $4x^2+3y^2+2ay=a^2$; (2) $x^2-3y^2+8ay=4a^2$.

V. (Pages 41, 42.)

1. $y=x+1$. **2.** $x-y-5=0$. **3.** $x-y\sqrt{3}-2\sqrt{3}=0$.
4. $5y-3x+15=0$. **5.** $2x+3y=6$. **6.** $6x-5y+30=0$.
7. (1) $x+y=11$; (2) $y-x=1$. **8.** $x+y+1=0$; $x-y=3$.
9. $xy'+x'y=2x'y'$. **10.** $20y-9x=96$. **15.** $x+y=0$.
16. $y-x=1$. **17.** $7y+10x=11$.
18. $ax-by=ab$. **19.** $(a-2b)x-by+b^2+2ab-a^2=0$.
20. $y(t_1+t_2)-2x=2at_1t_2$. **21.** $t_1t_2y+x=a(t_1+t_2)$.
22. $x\cos\frac{1}{2}(\phi_1+\phi_2)+y\sin\frac{1}{2}(\phi_1+\phi_2)=a\cos\frac{1}{2}(\phi_1-\phi_2)$.
23. $\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$.
24. $bx\cos\frac{1}{2}(\phi_1-\phi_2)-ay\sin\frac{1}{2}(\phi_1+\phi_2)=ab\cos\frac{1}{2}(\phi_1+\phi_2)$.
25. $x+3y+7=0$; $y-3x=1$; $y+7x=11$.
26. $2x-3y=4$; $y-3x=1$; $x+2y=2$.
27. $y(a'-a)-x(b'-b)=a'b-ab'$; $y(a'-a)+x(b'-b)=a'b'-ab$.
28. $2ay-2b'x=ab-a'b'$. **29.** $y=6x$; $2y=3x$.

VI. (Pages 48, 49.)

1. 90°. **2.** $\tan^{-1}\frac{23}{16}$. **3.** $\tan^{-1}\frac{4}{3}$. **4.** 60°.
5. $\tan^{-1}\frac{4m^2n^2}{m^4-n^4}$. **6.** $\tan^{-1}\frac{a^2-b^2}{2ab}$. **7.** $\tan^{-1}(2)$.
8. $4y+3x=18$. **9.** $7y-8x=118$. **10.** $4y+11x=10$.
11. $x+4y+16=0$. **12.** $ax+by=a^2$.
13. $2x(a-a')+2y(b-b')=a^2-a'^2+b^2-b'^2$.
15. $yx'-xy'=0$; $a^2xy'-b^2x'y=(a^2-b^2)x'y'$; $xx'-yy'=x'^2-y'^2$.
16. $121y-88x=371$; $33y-24x=1043$.
17. $x=3$; $y=4$; $4\frac{1}{2}$. **19.** $x=0$; $y+\sqrt{3}x=0$.
20. $y=k$; $(1-m^2)(y-k)=2m(x-h)$.
31. $\tan^{-1}\frac{13}{66}$; $9x-7y=1$; $7x+9y=73$.

VII. (Pages 53, 54.)

1. $4\frac{2}{5}$. **2.** $2\frac{3}{5}$. **3.** $5\frac{1}{13}$. **4.** $\frac{a^2+ab-b^2}{\sqrt{a^2+b^2}}$.

5. $a\cos\frac{1}{2}(\alpha-\beta)$. **8.** $\frac{c-d}{\sqrt{1+m^2}}$.

9. $\left\{\frac{a}{b}(b \pm \sqrt{a^2+b^2}),\ 0\right\}$. **11.** $\frac{1}{2}(2+\sqrt{3})$.

VIII. (Pages 61–65.)

1. $\left(\frac{-11}{29}, \frac{41}{29}\right)$. **2.** $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

3. $\left\{\frac{a}{m_1 m_2},\ a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}$.

4. $\{a\cos\frac{1}{2}(\phi_1+\phi_2)\sec\frac{1}{2}(\phi_1-\phi_2),\ a\sin\frac{1}{2}(\phi_1+\phi_2)\sec\frac{1}{2}(\phi_1-\phi_2)\}$.

5. $\left(\frac{a(b-b')}{b+b'}, \frac{2bb'}{b+b'}\right)$. **6.** $\frac{130}{17\sqrt{29}}$.

8. $y=a$; $3y=4x+3a$. **9.** $(1, 1)$; 45°.

10. $(\frac{5}{2}, \frac{1}{4})$; $\tan^{-1} 60$. **11.** $(-1, -3)$; $(3, 1)$; $(5, 3)$.

12. $(2, 1)$; $\tan^{-1}\frac{7}{17}$. **13.** 45°; $(-5, 3)$; $x-3y=9$; $2x-y=8$.

14. 3 and $-\frac{3}{2}$. **19.** $m_1(a_2-a_3)+m_2(a_3-a_1)+m_3(a_1-a_2)=0$.

20. $(-4, -3)$. **21.** $(\frac{11}{13}, -\frac{7}{13})$. **23.** $43x-29y=71$.

24. $x-y=11$. **25.** $y=3x$. **26.** $y=x$.

27. $a^2y-b^2x=ab(a-b)$. **28.** $3x+4y=5a$. **29.** $x+y+2=0$.

30. $23x+23y=11$. **31.** $13x-23y=64$.

33. $Ax+By+C+\lambda(A'x+B'y+C')=0$ where λ is

(1) $-\frac{C}{C'}$, (2) $-\frac{B}{B'}$, (3) $-\frac{Ba+C}{B'a+C'}$ and (4) $-\frac{Ax'+By'+C}{A'x'+B'y'+C'}$.

37. $y=2$; $x=6$. **38.** $99x+77y+71=0$; $7x-9y-37=0$.

39. $x-2y+1=0$; $2x+y=3$.

40. $x(2\sqrt{2}-3)+y(\sqrt{2}-1)=4\sqrt{2}-5$;

$x(2\sqrt{2}+3)+y(\sqrt{2}+1)=4\sqrt{2}+5$.

41. $(y-b)(m+m')+(x-a)(1-mm')=0$;

$(y-b)(1-mm')-(x-a)(m+m')=0$.

42. $33x+9y=31$; $112x-64y+141=0$; $7y-x=18$.

43. $x(3+\sqrt{17})+y(5+\sqrt{17})=15+4\sqrt{17}$;

$x(4+\sqrt{10})+y(2+\sqrt{10})=4\sqrt{10}+12$;

$x(2\sqrt{34}-3\sqrt{5})+y(\sqrt{34}-5\sqrt{5})=6\sqrt{34}-15\sqrt{5}$.

44. $A(y-k)-B(x-h)=\pm(Ax+By+C)$.

45. At an angle of 15° or 75° to the axis of x.

IX. (Pages 72, 73.)

1. (1) $\tan^{-1}\frac{\sqrt{3}}{2}$; (2) 15°. 2. $\tan^{-1}\frac{30\sqrt{3}}{37}$.

3. $\tan^{-1}\left(\frac{m^2+1}{m^2-1}\tan\omega\right)$.

7. $y=0$, $y=x-a$, $x=2a$, $y=2a$, $y=x+a$, $x=0$, $y=x$, $x=a$, and $y=a$, where a is the length of a side.

10. $y(6-\sqrt{3})+x(3\sqrt{3}-2)=22-9\sqrt{3}$. 11. $\frac{5}{8}$.

12. $10y-11x+1=0$; $\frac{8}{37}\sqrt{111}$.

X. (Pages 78—80.)

4. $(-7, 3)$. 5. $(-\frac{13}{24}, \frac{5}{4})$; $\frac{147}{16}$.

6. $\left(\frac{85-7\sqrt{5}}{120}, \frac{21\sqrt{5}-65}{120}\right)$; $\frac{35-7\sqrt{5}}{120}$. 7. $(\frac{1}{5}, \frac{14}{27})$; $\frac{14}{27}$.

8. $\left(\frac{6+\sqrt{10}}{2}, \frac{2+\sqrt{10}}{2}\right)$; $\left(\frac{6-\sqrt{10}}{2}, \frac{2-\sqrt{10}}{2}\right)$; $\left(\frac{8-\sqrt{10}}{6}, \frac{16+\sqrt{10}}{6}\right)$.

9. $(\frac{2}{3}, \frac{2}{3})$, $(2, 12)$, $(12, 2)$, and $(-3, -3)$; $\frac{2}{3}\sqrt{2}$, $1\sqrt{2}$, $4\sqrt{2}$, and $6\sqrt{2}$.

10. $(-13\frac{1}{2}, 19\frac{1}{2})$. 11. 4. 12. $7\frac{7}{18}$. 13. $\frac{3}{2}$.

14. $\frac{17a^2}{26}$. 15. $\frac{1}{2}(b-c)(c-a)(a-b)$.

16. $a^2(m_2-m_3)(m_3-m_1)(m_1-m_2)\div 2m_1^2m_2^2m_3^2$.

17. $\frac{1}{2}(c_1-c_2)^2\div(m_1-m_2)$. 18. $\frac{1}{2}\left\{\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}+\frac{(c_1-c_2)^2}{m_1-m_2}\right\}$.

23. $(\frac{5}{3}, \frac{5}{2})$.

24. $10y+32x+43=0$; $25x+29y+5=0$; $y=5x+2$; $52x+80y=47$.

26. $(4+\frac{1}{2}\sqrt{3}, \frac{3}{2}+\sqrt{3})$; $(4+\frac{1}{2}\sqrt{3}, \frac{3}{2}+\frac{1}{3}\sqrt{3})$.

XI. (Pages 85—87.)

1. $x^2+2xy\cot\alpha-y^2=a^2$. 2. $y^2+\lambda x^2=\lambda a^2$.

3. $(m+1)x=(m-1)a$. 4. $(m+n)(x^2+y^2+a^2)-2ax(m-n)=c^2$.

5. $x+y=c\sec^2\frac{\omega}{2}$. 6. $x-y=d\operatorname{cosec}^2\frac{\omega}{2}$.

7. $x+y=2c\operatorname{cosec}\omega$. 8. $y-x=2c\operatorname{cosec}\omega$.

9. $x^2+2xy\cos\omega+y^2=4c^2\operatorname{cosec}^2\omega$.

10. $(x^2+y^2)\cos\omega+xy(1+\cos^2\omega)=x(a\cos\omega+b)+y(b\cos\omega+a)$.

11. $x(m+\cos\omega)+y(1+m\cos\omega)=0$.

12. (i) $x+y-a-b=0$;
(ii) $y=x$. 19. A straight line.

20. A circle, centre O. 25. A straight line.

27. If P be the point (h, k), the equation to the locus of S is

$$\frac{h}{x}+\frac{k}{y}=1.$$

XII. (Page 94.)

1. $(x-3y)(x-4y)=0$; $\tan^{-1}\frac{1}{13}$. **2.** $(2x-11y)(2x-y)=0$; $\tan^{-1}\frac{4}{3}$.
3. $(11x+2y)(3x-7y)=0$; $\tan^{-1}\frac{83}{19}$. **4.** $x=1$; $x=2$; $x=3$.
5. $y=\pm 4$. **6.** $(y+4x)(y-2x)(y-3x)=0$; $\tan^{-1}(-\frac{6}{7})$; $\tan^{-1}(\frac{1}{7})$.
7. $x(1-\sin\theta)+y\cos\theta=0$; $x(1+\sin\theta)+y\cos\theta=0$; θ.
8. $y\sin\theta+x\cos\theta=\pm x\sqrt{\cos 2\theta}$; $\tan^{-1}(\operatorname{cosec}\theta\sqrt{\cos 2\theta})$.
9. $12x^2-7xy-12y^2=0$; $71x^2+94xy-71y^2=0$; $x^2-y^2=0$; $x^2-y^2=0$.

XIII. (Pages 98, 99.)

1. $(\frac{6}{5}, -\frac{12}{5})$; 45°. **2.** $(2, 1)$; $\tan^{-1}\frac{3}{5}$. **3.** $(-\frac{3}{2}, -\frac{5}{2})$; 90°.
4. $(-1, 1)$; $\tan^{-1}3$. **6.** -15. **7.** 2. **8.** -10 or $-17\frac{1}{2}$.
9. 12. **10.** 6. **11.** 6. **12.** 14. **13.** -3.
14. $\frac{4}{3}$ or $\frac{16}{3}$. **16.** (i) $c(a+b)=0$; (ii) $e=0$, or $ae=bd$.
17. $5y+6x=56$; $5y-6x=14$.

XV. (Page 112.)

1. (1) $y'^2=4x'$; (2) $2x'^2+y'^2=6$.
2. (1) $x'^2+y'^2=2cx'$; (2) $x'^2+y'^2=2cy'$.
3. $(a-b)^2(x'^2+y'^2)=a^2b^2$.
4. (1) $2x'y'+a^2=0$; $9x'^2+25y'^2=225$; $x'^4+y'^4=1$.
5. $x'^2+y'^2=r^2$; $x'^2-y'^2=a^2\cos 2a$. **6.** $x'^2-4y'^2=a^2$.
8. $\tan^{-1}\frac{B}{A}$; $-C\div\sqrt{A^2+B^2}$.

XVI. (Page 117.)

1. $2x'-\sqrt{6}y'+1=0$. **2.** $x'^2+\sqrt{3}x'y'=1$. **3.** $x'^2+y'^2=8$.
4. $y'^2=4x'\operatorname{cosec}^2 a$.

XVII. (Pages 123—125.)

1. $x^2+y^2+2x-4y=4$. **2.** $x^2+y^2+10x+12y=39$.
3. $x^2+y^2-2ax+2by=2ab$. **4.** $x^2+y^2+2ax+2by+2b^2=0$.
5. $(2, 4)$; $\sqrt{61}$. **6.** $(\frac{5}{6}, 1)$; $\frac{1}{6}\sqrt{13}$. **7.** $\left(\frac{k}{2}, 0\right)$; $\frac{\sqrt{5}}{2}k$.
8. $(g, -f)$; $\sqrt{f^2+g^2}$. **9.** $\left(-\frac{c}{\sqrt{1+m^2}}, -\frac{mc}{\sqrt{1+m^2}}\right)$; c.
13. $15x^2+15y^2-94x+18y+55=0$.
14. $b(x^2+y^2-a^2)=x(b^2+k^2-a^2)$. **15.** $x^2+y^2-ax-by=0$.
16. $x^2+y^2-22x-4y+25=0$. **17.** $x^2+y^2-5x-y+4=0$.
18. $3x^2+3y^2-29x-19y+56=0$.
19. $b(x^2+y^2)-(a^2+b^2)x+(a-b)(a^2+b^2)=0$.
21. $x^2+y^2-3x-4y=0$.

22. $x^2+y^2-\frac{a^2+b^2}{a+b}(x+y)=0$; $\frac{a^2+b^2}{a+b}$.

23. $x^2+y^2-hx-ky=0$.

24. $x^2+y^2\pm 2y\sqrt{a^2-b^2}=b^2$.

25. $x^2+y^2-10x-10y+25=0$.

26. $x^2+y^2\pm 2ax\pm 2ay+a^2=0$.

27. $x^2+y^2+2(5\pm\sqrt{12})(x+y)+37\pm 10\sqrt{12}=0$.

28. $x^2+y^2-6x+4y+9=0$, or $x^2+y^2+10x+20y+25=0$.

29. $b(x^2+y^2)=x(b^2+c^2)$.

30. $x^2+y^2\pm 6\sqrt{2}y-6x+9=0$.

31. $x^2+y^2-3x+2=0$; $2x^2+2y^2-5x-\sqrt{3}y+3=0$;
$2x^2+2y^2-7x-\sqrt{3}y+6=0$.

33. $(x+21)^2+(y+13)^2=65^2$.

34. $8x^2+8y^2-25x-3y+18=0$.

36. $x^2+y^2=a^2+b^2$; $x^2+y^2-2(a+b)x+2(a-b)y+a^2+b^2=0$.

XVIII. (Pages 134, 135.)

1. $5x-12y=152$.

2. $24x+10y+151=0$.

3. $x+2y=\pm 2\sqrt{5}$.

4. $x+2y+g+2f=\pm\sqrt{5}\sqrt{g^2+f^2-c}$.

5. $\left(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$.

6. $c=a$; $(0, b)$.

7. Yes.

8. $k=40$ or -10.

9. $a\cos^2\alpha+b\sin^2\alpha\pm\sqrt{a^2+b^2\sin^2\alpha}$.

10. $Aa+Bb+C=\pm c\sqrt{A^2+B^2}$.

11. (1) $y=mx\pm a\sqrt{1+m^2}$; (2) $my+x=\pm a\sqrt{1+m^2}$;
(3) $ax\pm y\sqrt{b^2-a^2}=ab$; (4) $x+y=a\sqrt{2}$.

12. $2\sqrt{r^2-\frac{a^2b^2}{a^2+b^2}}$.

13. $x^2+y^2\pm\sqrt{2}ax=0$; $x^2+y^2\pm\sqrt{2}ay=0$.

14. $c=b-am$; $c=b-am\pm\sqrt{(1+m^2)(a^2+b^2)}$.

15. $x^2+y^2-6x-8y+\frac{391}{169}=0$.

16. $x^2+y^2-2cx-2cy+c^2=0$, where $2c=a+b\pm\sqrt{a^2+b^2}$.

17. $5x^2+5y^2-10x+30y+49=0$.

18. $x^2+y^2-2cx-2cy+c^2=0$.

19. $(x-r)^2+(y-h)^2=r^2$.

20. $x^2+y^2-2\alpha x-2\beta y=0$.

XIX. (Pages 144, 145.)

1. $x+2y=7$.

2. $8x-2y=11$.

3. $x=0$.

4. $23x+5y=57$.

5. $by-ax=a^2$.

6. $(5, 10)$.

7. $(\frac{3}{5}, -\frac{3}{10})$.

8. $(1, -2)$.

9. $(\frac{1}{2}, -\frac{1}{3})$.

10. $(-2a, -2b)$.

11. $(6, -\frac{18}{5})$.

12. $3y-2x=13$; $(-\frac{162}{13}, \frac{243}{13})$.

13. $(2, -1)$.

14. $x'^2+y'^2=2a^2$.

18. $\frac{1}{2}\sqrt{46}$.

19. 9.

20. $\sqrt{2a^2+2ab+b^2}$.

21. $(\frac{23}{4}, 2)$; $\frac{1}{4}$.

23. (1) $28x^2+33xy-28y^2-715x-195y+4225=0$;
(2) $123x^2-64xy+3y^2-664x+226y+768=0$.

XX. (Pages 147, 148.)

1. $\left(\frac{1}{2}\sqrt{A^2+B^2},\ \tan^{-1}\frac{B}{A}\right)$.
2. $r^2-2ra\operatorname{cosec}\alpha\,.\cos(\theta-\alpha)+a^2\cot^2\alpha=0,\ r=2a\sin\theta$.
6. $r^2-r[a\cos(\theta-\alpha)+b\cos(\theta-\beta)]+ab\cos(\alpha-\beta)=0$.
8. $b^2c^2+2ac=1$.

XXI. (Page 149.)

1. $120°$; $\left(\frac{4g+2f}{3},\ \frac{4f+2g}{3}\right)$; $\frac{2\sqrt{3}}{3}\sqrt{f^2+g^2+fg}$.
2. $30°$; $(8-6\sqrt{3},\ 12-4\sqrt{3})$; $\sqrt{47-21\sqrt{3}}$.
3. $\left(\frac{g-f\cos\omega}{\sin^2\omega},\ \frac{f-g\cos\omega}{\sin^2\omega}\right)$; $\frac{\sqrt{f^2+g^2-2fg\cos\omega}}{\sin\omega}$
4. $x^2+\sqrt{2}xy+y^2-x(4+3\sqrt{2})-2y(3+\sqrt{2})+3(2\sqrt{2}-1)=0$.
5. $x^2+xy+y^2+11x+13y+13=0$.
8. $(x-x')(x-x'')+(y-y')(y-y'')+\cos\omega[(x-x')(y-y'')$
$+(x-x'')(y-y')]=0$.

XXII. (Pages 156—159.)

4. A circle. 5. A circle. 6. A circle.
9. $x^2+y^2-2xy\cos\omega=\frac{a^2\sin^2\omega}{4}$, the given radii being the axes.
11. A circle. 12. A circle.
16. (1) A circle; (2) A circle; (3) The polar of O.
17. The curve $r=a+a\cos\theta$, the fixed point O being the origin and the centre of the circle on the initial line.
24. The same circle in each case.
33. $2ab\div\sqrt{a^2+b^2}$. 35. $8a\sqrt{\frac{14}{35}}$; $x=4a$; $63x+16y+100a=0$.
36. (i) $x=0$, $3x+4y=10$, $y=4$, and $3y=4x$.
(ii) $y=mx+c\sqrt{1+m^2}$, where
$$m=\frac{\pm(b+c)}{\sqrt{a^2-(b+c)^2}},\ \text{or}\ \frac{\pm(b-c)}{\sqrt{a^2-(b-c)^2}}.$$

XXIII. (Pages 164, 165.)

3. $3x^2+3y^2-8x+29y=0$. 4. $15x-11y=144$.
5. $x+10y=2$. 6. $6x-7y+12=0$. 7. $(-\frac{2}{3},-\frac{2}{3})$.
8. $(\frac{32}{23},\frac{13}{30})$. 11. $(\lambda+1)(x^2+y^2)+2\lambda(x+2y)=4+6\lambda$.
12. $(y-x)^2=0$. 13. Take the equations to the circles as in Art. 192.

XXIV. (Pages 172, 173.)

8. $x^2 - y^2 + 2mxy = c$.

12. $k(x^2 + y^2) + (a - c)y - ck = 0$.

13. $x^2 + y^2 - cx - by + a^2 = 0$.

14. $x^2 + y^2 - 16x - 18y - 4 = 0$.

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